THE BIRMAN–MURAKAMI–WENZL ALGEBRAS OF TYPE D_n

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Abstract. The Birman–Murakami–Wenzl algebra (BMW algebra) of type D_n is shown to be semisimple and free over $\mathbb{Z}[\delta^{\pm 1}, l^{\pm 1}]/(m(1-\delta)-(l-l^{-1}))$ of rank $(2^n+1)n!! - (2^{n-1}+1)n!$, where $n!! = 1\cdot 3\cdots(2n-1)$. We also show it is a cellular algebra over suitable rings. The Brauer algebra of type D_n is a homomorphic ring image and is also semisimple and free of the same rank, but over the ring $\mathbb{Z}[\delta^{\pm 1}]$. A rewrite system for the Brauer algebra is used in bounding the rank of the BMW algebra above. As a consequence of our results, the generalized Temperley–Lieb algebra of type D_n turns out to be a subalgebra of the BMW algebra of the same type.

keywords: associative algebra, Birman–Murakami–Wenzl algebra, BMW algebra, Brauer algebra, cellular algebra, Coxeter group, generalized Temperley–Lieb algebra, root system, semisimple algebra, word problem in semigroups

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1. Introduction

In [1], Birman and Wenzl, and independently in [17], Murakami, defined algebras indexed by the natural numbers which play a role in both the representation theory of quantum groups and knot theory. They were given by generators and relations. In [16], Morton and Wasserman gave them a description in terms of tangles. These are the Birman–Murakami–Wenzl algebras for the Coxeter system of type A_{n-1}. They behave nicely with respect to restriction to the algebras generated by subsets of the generators. For instance, the BMW algebras of a restricted type embed naturally into the bigger ones. This is similar to the fact that in Weyl groups subgroups generated by subsets of the standard reflections are themselves Weyl groups. The Hecke algebra of type A_{n-1} is a natural quotient of the Birman–Murakami–Wenzl algebra (usually abbreviated to BMW algebra) of type A_{n-1} and the Temperley–Lieb algebra conceived originally for statistics (cf. [19]) is a natural subalgebra. Inspired by the beauty of these results, the existence of Temperley–Lieb algebras of other types ([12, 14, 9]) and the existence of a faithful linear representation of the braid group ([7, 8]), the authors defined analogues for other simply laced Coxeter diagrams and found some of their properties in [4].
In this paper we consider the algebras when the Coxeter diagram is of type D_n. We determine their rank as a free algebra and give some of their properties. In particular, we prove that the Birman–Murakami–Wenzl algebra of type D_n, as defined in [4], has rank $(2^n+1)n!! - (2^{n-1}+1)n!$, where $n!! = 1\cdot 3\cdots(2n-1)$. This settles the conjecture stated at the end of Section 7.1 in [4]. The proof requires an extension of the notion of a BMW algebra of simply laced type to integral

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rings of coefficients, see Definition 2.1 below. We work over the quotient ring R of $\mathbb{Z}[\delta, \delta^{-1}, l, l^{-1}, m]$ by the ideal generated by $m(1-\delta) - (l - l^{-1})$ instead of the field $\mathbb{Q}(l, \delta)$ in which it embeds (see Lemma 4.1).

Theorem 1.1. *The BMW algebra of type* D_n *over* R *is free of rank*

$$(2^n + 1)n!! - (2^{n-1} + 1)n!.$$

When tensored with $\mathbb{Q}(l, \delta)$*, it is semisimple.*

The result produces linear representations of the Artin group of type D_n similar to the representations of the braid group of n strands which arose from the BMW algebra of type A_{n-1}. These include the faithful representations related to the Laurence–Krammer representations occurring in [7] as well as the representations occurring in [4]. Furthermore, specific information about the representations is given in terms of sets of orthogonal roots and irreducible representations of Weyl groups of type D_r for certain r. We also show that, for suitable extensions of the ring R, the BMW algebra $\mathbf{B}(\mathrm{D}_n)$ is cellular in the sense of [13, Definition 1.1]. For $\mathbf{B}(\mathrm{A}_n)$, this result is known thanks to [21].

Theorem 1.2. *The BMW algebra of type* D_n *is cellular provided the coefficient ring* R *is extended to an integral domain containing an inverse to* 2.

We proceed as follows. First, in Section 2, we introduce the BMW algebra $\mathbf{B}(M)$ over R for M of type A_n ($n \geq 1$), D_n ($n \geq 4$), or E_n ($n = 6, 7, 8$), which we denote ADE. Then, the Brauer algebra, $\mathbf{Br}(M)$, of the same type over $\mathbb{Z}[\delta^{\pm 1}]$ is obtained by taking the quotient by the ideal generated by m and $l - 1$. This algebra was defined in [3] where it was shown to be free over R of rank $(2^n+1)n!! - (2^{n-1}+1)n!$ in case $M = \mathrm{D}_n$. The modding out of m and $l-1$ gives a surjective homomorphism $\mu : \mathbf{B}(M) \mapsto \mathbf{Br}(M)$. The Brauer algebra $\mathbf{Br}(M)$ is given in terms of generators e_i, r_i for i running over the nodes of M, and relations determined by M. The subalgebra of $\mathbf{Br}(M)$ generated by the r_i is the group algebra over $\mathbb{Z}[\delta^{\pm 1}]$ of $W(M)$, the Coxeter group of type M. At the end of Section 2 and at the beginning of Section 3, we summarize results from [3] which show how to represent the monomials of $\mathbf{Br}(M)$ by certain sets of orthogonal roots, which in the case $M = \mathrm{A}_{n-1}$ are directly related to the well-known Brauer diagrams. The monomials, including powers of δ, form a monoid inside $\mathbf{Br}(M)$, denoted $\mathbf{BrM}(M)$ (see Definition 2.3). In Section 3 we use the following strategy to exhibit a spanning set of $\mathbf{B}(\mathrm{D}_n)$ consisting of elements of $\mathbf{BrM}(\mathrm{D}_n)$. A word $\underline{a}$ in the generators of the Brauer monoid $\mathbf{BrM}(M)$ is said to be of height t if the number of generators r_i occurring in it is equal to t. We say that $\underline{a}$ is reducible to another word $\underline{b}$ if $\underline{b}$ can be obtained from $\underline{a}$ by a sequence of specified rewrites (listed in Table 2), that do not increase the height. This process will be called a reduction. The significance of such a reduction is that the word $\underline{a}$ also corresponds to a unique monomial in the BMW algebra and that a parallel reduction (with rules listed in Table 1) can be carried out in the BMW algebra in the sense that the monomial in $\mathbf{B}(\mathrm{D}_n)$ corresponding to $\underline{a}$ can be rewritten as a linear combination of monomials all of which are represented by words of height less than or equal to the height of $\underline{a}$, with equality occurring for at most one term (see Proposition 2.5(ii)). For a relevant part of $\mathbf{B}(\mathrm{D}_n)$ we exhibit a finite set of reduced words to which each word reduces, see Corollary 3.18. This will lead to a set T of reduced words such that every word in the generators of

$\mathbf{B}(\mathrm{D}_n)$ can be reduced to an element of T. The above argument will give that, when viewed as elements of $\mathbf{B}(\mathrm{D}_n)$, the set T is a spanning set of $\mathbf{B}(\mathrm{D}_n)$.

We conjecture that similar reductions will also work in types E_6, E_7, and E_8, so that the ranks of the BMW algebras will be the same as the ranks of the Brauer algebras of the corresponding type to be found in [3, Theorem 1.1]. The generalized Temperley–Lieb algebras of type E_n have finite rank for all $n \geq 6$, as proved in [12] (see also [9]), but the Brauer algebras have only finite rank for types E_6, E_7, and E_8.

In Section 4 we show how to specialize in R to enable us to pass from elements of T in $\mathbf{B}(\mathrm{D}_n)$ to monomials in $\mathbf{Br}(\mathrm{D}_n)$ and infer that they are linearly independent in $\mathbf{B}(\mathrm{D}_n)$. In Section 5 we prove our main result by considering parts of T corresponding to various ideals in both $\mathbf{B}(\mathrm{D}_n)$ and $\mathbf{Br}(\mathrm{D}_n)$. We also observe that the generalized Temperley–Lieb algebra of type D_n, as defined in [9, 12, 14] embeds in $\mathbf{B}(\mathrm{D}_n)$ and in $\mathbf{Br}(\mathrm{D}_n)$.

In Section 6 we show that if the ring of coefficients is extended to an integral domain containing 2^{-1}, the algebra $\mathbf{B}(\mathrm{D}_n)$ is cellular in the sense of [13, Definition 1.1]. The ring extension is necessary in order to use [11, Theorem 1.1] where cellularity of the Hecke algebras of type D_n is proved for such rings of coefficients. This is needed in our work as this Hecke algebra is a natural quotient and the Hecke algebras of type D_{n-2t} occur as subalgebras with different idempotents as identities in the analysis.

We will apply the above results in [6], where a tangle algebra $\mathbf{KT}(\mathrm{D}_n)$ over R on n strands is introduced. It is shown to be a homomorphic image of the BMW algebra of type D_n and Theorem 1.1 will be used to infer that it is an isomorphic image.

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2. BMW and Brauer Algebras

The BMW algebras of type A_n ($n \geq 1$), D_n ($n \geq 4$), and E_n ($n = 6, 7, 8$) have been discussed extensively in [4]. We assume that M is a Coxeter diagram which is one of these (in particular, it has no multiple bonds). Our main results will only concern M of type A_{n-1} and D_n. The BMW algebra of type M is defined over the ring $R = \mathbb{Z}[l^{\pm 1}, m, \delta^{\pm 1}]/(m(\delta - 1) - (l^{-1} - l))$.

Definition 2.1. The BMW algebra $\mathbf{B}(M)$ of type M is the free algebra over R given by generators g_i, e_i with i running over the nodes of the diagram M, subject to the relations in the BMW Relations Table 1 where $i \sim j$ denotes adjacency of two nodes i and j.

Remark 2.2. The set of relations given is superfluous. In fact, the relations (HNrer), (HNree), (HNeer), (HNeee), (HTeere), and (RTerre) are not needed, as we now explain. Moreover, if $\mathbf{B}(M)$ is tensored with a ring in which m is invertible, then only the relations (RSrr), (RSre), (HCrr), (HNrrr), and (RNere) are needed; in [4] these were labelled (D1), (R1), (B1), (B2), and (R2), respectively.

We will prove these dependencies, starting with (HNeee). By (RNrre), (RSre), and (RNere), respectively, we have

$$e_ie_je_i \quad = \quad e_ig_jg_ie_i = e_ig_je_il^{-1} = e_i.$$

	for i
(RSrr)	$g_i^2 = 1 - m(g_i - l^{-1}e_i)$
(RSer)	$e_i g_i = l^{-1} e_i$
(RSre)	$g_i e_i = l^{-1} e_i$
(HSee)	$e_i^2 = \delta e_i$
	for $i \not\sim j$
(HCrr)	$g_i g_j = g_j g_i$
(HCer)	$e_i g_j = g_j e_i$
(HCee)	$e_i e_j = e_j e_i$
	for $i \sim j$
(HNrrr)	$g_i g_j g_i = g_j g_i g_j$
(HNrer)	$g_j e_i g_j = g_i e_j g_i + m(e_j g_i - e_i g_j + g_i e_j - g_j e_i)$ $+m^2(e_j - e_i)$
(RNrre)	$g_j g_i e_j = e_i e_j$
(RNerr)	$e_i g_j g_i = e_i e_j$
(HNree)	$g_j e_i e_j = g_i e_j + m(e_j - e_i e_j)$
(RNere)	$e_i g_j e_i = l e_i$
(HNeer)	$e_j e_i g_j = e_j g_i + m(e_j - e_j e_i)$
(HNeee)	$e_i e_j e_i = e_i$
	for $i \sim j \sim k$
(HTeere)	$e_j e_i g_k e_j = e_j g_i e_k e_j$
(RTerre)	$e_j g_i g_k e_j = e_j e_i e_k e_j + m(e_j e_i g_k e_j - l e_j)$

TABLE 1. BMW Relations Table

For (HNeer) we multiply (RNerr) by g_j and apply (RSrr), and, for the final equality, (RNerr), and (RNere):

$$\begin{aligned} e_j e_i g_j &= e_j g_i g_j^2 = e_j g_i(1 - m g_j + m l^{-1} e_j) = e_j g_i - m e_j g_i g_j + m l^{-1} e_j g_i e_j \\ &= e_j g_i - m e_j e_i + m e_j. \end{aligned}$$

(HNree) is derived in a similar way. The equation (HNrer) is dealt with in [4, Proposition 2.3] by use of the relations we have obtained. For (HTeere), we use (RNerr) and (RNrre), respectively:

$$e_j e_i g_k e_j = e_j g_i g_j g_k e_j = e_j g_i e_k e_j.$$

Recall here that $i \not\sim k$ because the diagram M has no triangles. For (RTerre) write $e_j g_i g_k e_j = e_j g_i g_j g_j^{-1} g_k e_j$ and use the expression for g_j^{-1} from (RSrr). This is also in [4, Proposition 2.1].

Definitions 2.3. Let M be a graph of type ADE. We define the Brauer monoid $\mathbf{BrM}(M)$ to be the monoid generated by the elements r_i and e_i ($i \in M$) and δ subject to the relations in the Brauer Relations Table 2. The Brauer algebra of type M is the monoid algebra $\mathbb{Z}[\mathbf{BrM}(M)]$.
The r_i in $\mathbf{BrM}(M)$ generate a subgroup of the Brauer monoid that we denote W. This is a Coxeter group of type M as the r_i satisfy the required relations and, after factoring out the ideal generated by the e_i and $\delta - 1$, we obtain the standard presentation of this Coxeter group by generators and relations.
The Brauer algebra of type M is really an algebra over $\mathbb{Z}[\delta^{\pm 1}]$ as δ is in the center of $\mathbf{BrM}(M)$. Since the other defining relations of the Brauer monoid are the defining

relations of the corresponding BMW algebra $\mathbf{B}(M)$ modulo the ideal $(l-1,m)$ generated by $l-1$ and m, the Brauer algebra $\mathbf{BrM}(M)$ can be identified with $\mathbf{B}(M)\otimes_R R/(l-1,m)$. The corresponding map $a\mapsto a\otimes 1$ will be denoted by μ.

label	relation	label	relation
(δ)	δ is central	(δ^{-1})	$\delta\delta^{-1}=1$
	for i		
(RSrr)	$r_i^2=1$	(RSer)	$e_ir_i=e_i$
(RSre)	$r_ie_i=e_i$	(HSee)	$e_i^2=\delta e_i$
	for $i\not\sim j$		
(HCrr)	$r_ir_j=r_jr_i$	(HCer)	$e_ir_j=r_je_i$
(HCee)	$e_ie_j=e_je_i$		
	for $i\sim j$		
(HNrrr)	$r_ir_jr_i=r_jr_ir_j$	(HNrer)	$r_je_ir_j=r_ie_jr_i$
(RNrre)	$r_jr_ie_j=e_ie_j$	(RNerr)	$e_ir_jr_i=e_ie_j$
(HNree)	$r_je_ie_j=r_ie_j$	(RNere)	$e_ir_je_i=e_i$
(HNeer)	$e_je_ir_j=e_jr_i$	(HNeee)	$e_ie_je_i=e_i$
	for $i\sim j\sim k$		
(HTeere)	$e_je_ir_ke_j=e_jr_ie_ke_j$	(RTerre)	$e_jr_ir_ke_j=e_je_ie_ke_j$

Table 2. Brauer Relations Table

Just as for $\mathbf{B}(M)$, there are more relations than needed in the Brauer Relations Table 2; see [3, Lemma 3.1]. We are interested in ways to rewrite words in the generators r_i and e_i, with $\delta^{\pm1}$ viewed as coefficients.

Definitions 2.4. By F_n we denote the monoid that is the central product of the free monoid on the symbols r_i, e_i $(i=1,\ldots,n)$ with the infinite cyclic group generated by δ. Its elements will be called words. There is a surjective homomorphism of monoids $\pi: F_n\to\mathbf{BrM}(M)$ mapping the symbols r_i, e_i, and δ to the corresponding elements of $\mathbf{BrM}(M)$. The monomial in $\mathbf{B}(M)$ corresponding to $\underline{a}\in F_n$, obtained by replacing r_i by g_i and leaving e_i and δ as before, will be denoted $\rho(\underline{a})$, so $\mu(\rho(\underline{a}))=\pi(\underline{a})$. For a word $\underline{a}$ in F_n we say that $\pi(\underline{a})$ is a monomial in the Brauer monoid and $\rho(\underline{a})$ is a monomial in the BMW algebra. A word $\underline{a}\in F_n$ is said to be of *height* t if the number of r_i occurring in it is equal to t; we denote this number t by $\mathrm{ht}(\underline{a})$.

We say that $\underline{a}$ is *reducible* to another word $\underline{b}$, or that $\underline{b}$ is a reduction of $\underline{a}$, if $\underline{b}$ can be obtained by a sequence of specified rewrites, listed in the Brauer Relations Table 2, starting from $\underline{a}$, that do not increase the height. We call a word in F_n *reduced* if it cannot be further reduced to a word of smaller height. We have labelled the relations in the tables above with R or H according to whether the rewrite from left to right strictly lowers the height or not. If the number stays the same, we call it H for homogeneous. Our rewrite system will be the set of all rewrites in the Brauer Relations Table 2 from left to right and vice versa in the homogeneous case and from left to right in case an R occurs in its label. We write $\underline{a}\rightsquigarrow\underline{b}$ if $\underline{a}$ can be reduced to $\underline{b}$; for example (RNerr) gives $e_2r_3e_2\rightsquigarrow e_2$ if $2\sim 3$. If the height does not decrease during a reduction, we sometimes use the term *homogeneous reduction*

and write $\underline{a} \leftrightsquigarrow \underline{b}$; for example, (HNeee) gives $e_2 \leftrightsquigarrow e_2e_3e_2$ if $2 \sim 3$. If it does decrease, we also speak of a *strict reduction*.
Homogeneous reduction induces an equivalence relation on F_n, to which we will refer as *homogeneous equivalence*. We denote its set of equivalence classes by $F_n/\leftrightsquigarrow$.

The reductions in F_n are important because they have a meaning for both the Brauer algebra and the corresponding BMW algebra. For each of the relations in the Brauer Relations Table 2, there is a corresponding relation in the BMW Relations Table 1. In Section 5, the following proposition will be used to find a basis of $\mathbf{B}(\mathrm{D}_n)$ that has the same size as a basis of $\mathbf{Br}(\mathrm{D}_n)$.

Proposition 2.5. *Suppose $\underline{a} \rightsquigarrow \underline{b}$ with $\underline{a}$, $\underline{b} \in F_n$.*
(i) $\pi(\underline{a}) = \pi(\underline{b})$ in $\mathbf{BrM}(M)$.
(ii) There are $\lambda_c \in R$ such that $\rho(\underline{a}) = \rho(\underline{b}) + \sum_{\underline{c}\in F_n,\, \mathrm{ht}(\underline{c})<\mathrm{ht}(\underline{a})} m\lambda_c\rho(\underline{c})$ in $\mathbf{B}(M)$.

Proof. (i). For each of the sequence of relations, the word evaluated in $\mathbf{BrM}(M)$ is the same because the relations are satisfied in $\mathbf{BrM}(M)$ by definition. This means $\pi(\underline{a}) = \pi(\underline{b})$. This proves (i).
(ii). The expressions in the BMW Relations Table 1 all have one term on each side whose coefficient is not a multiple of m. These terms are the same as in the Brauer Relations Table 2 with g_i instead of r_i. Indeed, if $l = 1$ and the terms with coefficient m are ignored, the tables are the same. Each iteration in $\underline{a} \rightsquigarrow \underline{b}$, replaces the term without coefficient m with the corresponding one on the other side of the equality in the table plus terms that are multiples of m and have smaller height. The end result is $\rho(\underline{b})$ plus terms that are multiples of m, whose height has been reduced at least once. $\square$

In the remainder of this section we summarize some of the results of [5] and [3] about admissible sets. These are particular sets of mutually orthogonal positive roots. The results will be used to monitor the reduction of words for $M = \mathrm{D}_n$. We will fix a root system Φ for W and a set of simple roots $\alpha_1, \ldots, \alpha_n$ with indices for $M = \mathrm{D}_n$ as indicated in the Dynkin diagram of Figure 1. In terms of the standard orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$, these simple roots are $\alpha_1 = \varepsilon_1 + \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_1$, $\alpha_3 = \varepsilon_3 - \varepsilon_2$, …, $\alpha_n = \varepsilon_n - \varepsilon_{n-1}$. Accordingly, we will write $r_i = r_{\alpha_i}$ and $\Phi^+ = (\mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \cdots + \mathbb{Z}_{\geq 0}\alpha_n) \cap \Phi$ where $\mathbb{Z}_{\geq 0}$ are the non-negative integers. The elements of Φ^+ are called the positive roots of D_n; they are of the form $\varepsilon_j - \varepsilon_i$ and $\varepsilon_i + \varepsilon_j$ for $n \geq j > i \geq 1$. Recall $\Phi = \Phi^+ \cup (-\Phi^+)$.

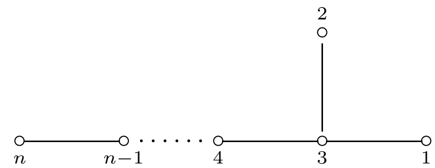

FIGURE 1. The diagram of type D_n with node labels

In order to recognize the elements of the ideal Θ in $\mathbf{BrM}(\mathrm{D}_n)$ generated by e_1e_2, we will use the notion of orthogonal mates.

Definition 2.6. For $\beta = \varepsilon_i - \varepsilon_j$ a root in the root system of Φ type D_n embedded in $\mathbb{R}^n$ as indicated above, its *orthogonal mate* is defined to be $\beta^* = \varepsilon_i + \varepsilon_j$ and, vice versa, the *orthogonal mate* of β^* is $\beta^{**} = \beta$. Furthermore, we write r^*_β for r_{β^*}, the reflection whose root is the orthogonal mate of β. For the simple roots α_i we let $r^*_{\alpha_i}$ be denoted r^*_i.

If $n > 4$, the set of roots orthogonal to β form a subsystem of Φ type $\mathrm{A}_1\mathrm{D}_{n-2}$ and β^* is the unique positive root in the A_1 component of this subsystem. If $n = 4$, the choice of orthogonal mate essentially depends on the choice of fundamental roots.

There are several equivalent definitions of admissible sets as outlined in [5, Proposition 2.3]. A representative of each orbit of admissible sets is given in [3, Table 3]; this is a corrected version of a similar table in [5]. We will need these only for types A_n and D_n. For our purposes we may define a set S of mutually orthogonal positive roots to be admissible if and only if, when $\alpha_1, \alpha_2, \alpha_3 \in S$ and there exists a root α for which $(\alpha_i, \alpha) = \pm 1$ for all i, then $r_\alpha r_{\alpha_1} r_{\alpha_2} r_{\alpha_3}\alpha$ or $-r_\alpha r_{\alpha_1} r_{\alpha_2} r_{\alpha_3}\alpha$ is also in S. Given any set, S, of mutually orthogonal positive roots, a straightforward exercise shows there is a unique smallest admissible set containing S. This set is called the *admissible closure* of S, notation S^{cl}; see [3]. The Weyl group W of type M acts on S by means of $wS = \{\pm ws \mid s \in S\} \cap \Phi^+$ for $w \in W$.
By $\mathcal{A}$ we denote the disjoint union of all admissible W-orbits (including the empty set). In [5] it is shown that there is a natural order on each W-orbit in $\mathcal{A}$, and in fact each such orbit has a unique maximal element under this order. The following proposition is proved in [3, Theorem 3.6]; the fact that e_iB as described below is well defined is shown in [3, Lemma 3.3(v)].

Proposition 2.7. *The action of W on $\mathcal{A}$ extends to an action of the Brauer monoid determined on the generators by*

$$\begin{aligned} \delta B &= B, \\ e_i B &= \begin{cases} B & \text{if } \alpha_i \in B, \\ (B \cup \{\alpha_i\})^{cl} & \text{if } \alpha_i \perp B, \\ r_\beta r_i B & \text{if } \beta \in B \setminus \alpha_i^\perp \end{cases} \end{aligned}$$

for $i \in M$ and $B \in \mathcal{A}$. For this action, if X, $Y \in \mathcal{A}$ and $a \in \mathbf{BrM}(M)$ satisfy $X \subseteq Y$, then $aX \subseteq aY$.

The only admissible sets we will need for D_n are $Y(t) = \{\alpha_n, \alpha_{n-2}, \ldots, \alpha_{n-2t+2}\}$ of size t with $Y(0) = \emptyset$, and if $n = 2t$, also $Y'(n/2) = \{\alpha_n, \alpha_{n-2}, \ldots, \alpha_4, \alpha_1\}$. The extra possibility $Y'(n/2)$ when $n = 2t$ is in a different orbit than $Y(n/2)$, as shown in [3, Lemma 1.2]. Notice that, if β is in $Y(t)$, then β^* is not in it. The orbits of orthogonal roots treated in [3, Table 3] consist of these $Y(t)$ in the first line under D_n and the sets $Y(t)$ together with their orthogonal mates which appear under the second line. The following result shows that the latter sets lead to elements of Θ, the ideal of $\mathbf{BrM}(\mathrm{D}_n)$ generated by e_1e_2.

Lemma 2.8. *An element $a \in \mathbf{BrM}(\mathrm{D}_n)$ belongs to the ideal Θ generated by e_1e_2 if and only if $a(\emptyset)$ contains a pair β, β^* for some $\beta \in \Phi^+$.*

Proof. See [3, Table 3 and Proposition 4.9]. □

Remark 2.9. There are two notions of height. The first is $\mathrm{ht}(\underline{a})$ for $\underline{a}$ an element of F_n. This is the number of r_i appearing in the monomial $\underline{a}$. The other is the more standard notion of height of a positive root β. This is $\sum c_i$ for the root $\sum c_i\alpha_i$ where the α_i are the simple roots. We also denote this $\mathrm{ht}(\beta)$ and trust no confusion will arise.

3. Reduction in the Brauer monoid

In this section we show how to reduce words in F_n for the Brauer monoid, $\mathbf{BrM}(\mathrm{D}_n)$, of type D_n. We will set aside the words in F_n whose images under π lie in the ideal Θ of $\mathbf{BrM}(\mathrm{D}_n)$ generated by e_1e_2. The reason is that Θ can be dealt with by using the more familiar case of type A_{n-1}. The purpose of this section is to show that, up to homogeneous equivalence, each element of $F_n \setminus \pi^{-1}\Theta$ has a unique reduced word. This goal is achieved in Corollary 3.18.

We will use the action of Proposition 2.7. Let $\underline{a} \in F_n\setminus\pi^{-1}(\Theta)$. Then, by Lemma 2.8 and [3, Table 3], $\pi(\underline{a})(\emptyset) \in WY(t)$ up to an interchange of 1 and 2 in the case where $t = n/2 \in \mathbb{N}$. The interchange of the two nodes is justified as it corresponds to an automorphism of D_n and hence also of $\mathbf{BrM}(\mathrm{D}_n)$. This way we do not need to treat the case $Y'(n/2)$ separately.

Notation 3.1. In [3, Section 4] elements e_X are defined for $X \in \mathcal{A}$. We will adopt this notation so, for $X = Y(t)$, we will have $e_{Y(t)} = e_ne_{n-2}\cdots e_{n-2t+2}$. All factors commute, so we need not care about the order in which they occur.
Recall that the subgroup W of $\mathbf{BrM}(\mathrm{D}_n)$ generated by $r_1,\ldots,r_n$ is the Coxeter group of type D_n. For $X \subseteq \Phi$ we let $N_W(X)$ be the normalizer of X in W. Let D_X be a set of left coset representatives of $N_W(X)$ in W and let C_{WX} be the set of nodes in M whose corresponding roots are orthogonal to all members of the unique maximal element in the W-orbit WX within the poset $\mathcal{A}$ (as discussed above Proposition 2.7). For $x_1,\ldots,x_q \in \{r_1,\ldots,r_n,e_1,\ldots,e_n,\delta^{\pm1}\}$, we write $(x_1\cdots x_q)^{\mathrm{op}} = x_q\cdots x_1$, thus defining an opposition map on F_n. This notation is compatible with the maps π and ρ when $\cdot^{\mathrm{op}}$ on $\mathbf{B}(\mathrm{D}_n)$ and $\mathbf{Br}(\mathrm{D}_n)$ is interpreted as the anti-involution of [4] and [3], respectively.

We recall from [3, Table 3] that

$$C_{WY(t)} = \begin{cases} \mathrm{D}_n & \text{if } t=0\\ \mathrm{A}_1\mathrm{D}_{n-2t} & \text{if } 0<t<(n-1)/2\\ \mathrm{A}_1 & \text{if } t=(n-1)/2\\ \emptyset & \text{if } t=n/2\end{cases} \tag{1}$$

and $C_{WY'(n/2)} = \emptyset$. The following theorem is proved in [3, Proposition 4.9].

Theorem 3.2. *Let $M = \mathrm{D}_n$ and suppose that $\underline{a}$ is a word in F_n. Let X be the unique maximal element of a W-orbit in $\mathcal{A}$. Then $\pi(\underline{a})$ is of the form $ue_Xzv^{\mathrm{op}}\delta^k$ for some $k \in \mathbb{N}$, and $z \in W(C_{WX})$ with $u, v \in D_X$ such that $uX = \pi(a)(\emptyset)$ and $vX = \pi(\underline{a}^{\mathrm{op}})(\emptyset)$.*

We will prove a counterpart of this result where we consider only elements not in Θ. The role of the maximal element X in the W-orbit in $\mathcal{A}$ will be taken up by $Y(t)$.

In fact, we will prove a stronger statement, about rewrites of $\underline{a}$ in F_n rather than equality for $\pi(\underline{a})$. This will be Corollary 3.18 below. In general, the expressions $ue_{Y(t)}zv^{\mathrm{op}}\delta^k$ of Theorem 3.2 are not reduced, which makes them unsuitable for rewrite purposes. We will be concerned with replacing e_X, u, v, and z by words of lower height. To appreciate the need for an alternative to the maximal element X in $WY(t)$ in order to work with reduction, notice that $e_{Y(t)}$ has height 0, whereas e_X in general does not. Also, by way of example, we mention that, for $e_n r_n^*$, we will find an expression of height 1.

Our immediate goal will be to to show that $\underline{a}$ can be rewritten to a reduced word that is uniquely determined by $\pi(\underline{a})$ up to homogeneous equivalence, so the reduced word will be a unique element of $F_n/\leftrightsquigarrow$. In fact, we shall be working with words in F_n but often think of them as representing classes in $F_n/\leftrightsquigarrow$. The rewrite version is of importance in finding an upper bound for the rank of $\mathbf{B}(\mathrm{D}_n)$ in the proof of Theorem 1.1.

Later, in Section 6, we will use words in F_n to represent monomials in $\mathbf{B}(\mathrm{D}_n)$. Then a reduced word $\underline{a}$ will not always be uniquely determined by $\rho(\underline{a})$ unless $\underline{a}$ has all of its symbols in $\{r_1, \dots, r_n, \delta\}$.

Before we continue we introduce some notation. Suppose that k and i are two nodes of D_n. Let $i = i_1, i_2, \dots, i_r = k$ be the geodesic path from i to k in D_n. Then we set $e_{i,k} = e_{i_1}e_{i_2}e_{i_3}\cdots e_{i_r}$, which we interpret as an element of F_n. Notice the first factor is e_i and the last is e_k. In particular, for $i < k$ and $k \geq 3$, we have $e_{i,k} = e_i e_{i+1}\cdots e_k$ unless $i = 1$ in which case it is $e_1e_3e_4\cdots e_k$. Also $e_{1,2} = e_1e_3e_2$ is a special case.

Let β be a positive root. Recall from [4] that the support of β is the set of nodes whose corresponding simple roots occur in an expression of β as a sum of simple roots; it is denoted $\mathrm{Supp}(\beta)$. As in [4], we will write, for k a node of the diagram, $\mathrm{Proj}(k, \beta)$ for the node of D_n in $\mathrm{Supp}(\beta)$ nearest to k. There is a unique one as the support is a connected set of nodes in the Dynkin diagram.

Definition 3.3. If $k \in \mathrm{Supp}(\beta)$, then, as follows directly from [4, Proposition 2.3], there is a unique Weyl group element $a_{\beta,k}$ of smallest length that maps $\{\alpha_k\}$ to $\{\beta\}$ in the action of Proposition 2.7 (so $a_{\beta,k}\alpha_k = \beta$). Its height, as a monomial of $\mathbf{Br}(\mathrm{D}_n)$ is equal to $\mathrm{ht}(\beta)-1$. The opposite element $a_{\beta,k}^{\mathrm{op}}$ maps $\{\beta\}$ to $\{\alpha_k\}$. We will often view $a_{\beta,k}$ as an element of F_n in the guise of a shortest expression for $a_{\beta,k}$ as a product of simple reflections. Since any two such expressions are homogeneously equivalent, they represent the same element of $F_n/\leftrightsquigarrow$, which suffices for our purpose of reductions.

We extend the definition of $a_{\beta,k}$ to the case where $k \notin \mathrm{Supp}(\beta)$. For β a positive root with $k \notin \mathrm{Supp}(\beta)$ and k' the node next to k on the geodesic path from k to $j = \mathrm{Proj}(k, \beta)$, we set $a_{\beta,k} = a_{\beta,j}e_{j,k'}$ in F_n.

We will be mainly concerned with the case $k = n$.

Lemma 3.4. *The elements $a_{\beta,n}$ satisfy the following properties.*

(i) If $j \leq n-1$, then $a_{\alpha_j,n}e_n = e_{j,n}$.
(ii) If j is a node of D_n such that $\beta - \alpha_j$ is a root, then $a_{\beta,n}e_n \leftrightsquigarrow r_j a_{\beta-\alpha_j,n}e_n$.

Proof. (i). Clearly n is not in the support of α_j and the identity maps α_j to a fundamental root. In particular $a_{\alpha_j,n} = e_{j,n-1}$ and $a_{\alpha_j,n}e_n = e_{j,n}$.

(ii). We first consider the case where $n \in \mathrm{Supp}(\beta)$. In this case $a_{\beta,n}$ is any word of shortest length which takes α_n to β. Its length is $\mathrm{ht}(\beta)-1$. As mentioned above and in [4, Proposition 2.3] there is a unique one up to homogeneous equivalence. If $j \neq n$, then $a_{\beta-\alpha_j,\alpha_n}$ is a word of shortest length taking α_n to $\beta - \alpha_j$ and so $r_j a_{\beta-\alpha_j,\alpha_n}$ is a word of shortest length taking α_n to β, proving that $a_{\beta,n} \leftrightsquigarrow r_j a_{\beta-\alpha_j,n}$, and so $a_{\beta,n}e_n \leftrightsquigarrow r_j a_{\beta-\alpha_j,n}e_n$.
If $j = n$, then $\beta - \alpha_n$ is a root. Because of the structure of the roots of D_n, this means the coefficient in β of both α_n and α_{n-1} as a linear combination of simple roots is 1 and so $\beta - \alpha_n$ has $n-1$ in its support but not n. In particular, $a_{\beta-\alpha_n,n-1}$ is a word of height $\mathrm{ht}(\beta) - 1$ taking α_{n-1} to $\beta - \alpha_n$. As n is not in the support of $\beta - \alpha_n$ but $n-1$ is, $a_{\beta-\alpha_n,n-1}$ is a word in r_j with $j \leq n-1$ and further as the coefficient of α_{n-1} in $\beta - \alpha_n$ is just 1, all the r_j occurring in a reduced decomposition of $a_{\beta-\alpha_n,n-1}$ have j at most $n-2$. In particular r_n and $a_{\beta-\alpha_n,n-1}$ commute. Also, $a_{\beta-\alpha_n,n} = a_{\beta-\alpha_n,n-1}e_{n-1}$ by definition. Now $r_n a_{\beta-\alpha_n,n}e_n = r_n a_{\beta-\alpha_n,n-1}e_{n-1}e_n \leftrightsquigarrow a_{\beta-\alpha_n,n-1}r_n e_{n-1}e_n$. By (HNree) $r_n e_{n-1}e_n \leftrightsquigarrow r_{n-1}e_n$. In terms of the action of Proposition 2.7, this implies $a_{\beta-\alpha_n,n-1}r_{n-1}\{\alpha_n\} = a_{\beta-\alpha_n,n-1}\{\alpha_{n-1}+\alpha_n\}$. Recall $a_{\beta-\alpha_n,n-1}$ is a shortest word in $r_1,\ldots,r_{n-1}$ taking α_{n-1} to $\beta - \alpha_n$ and so is a word of shortest length taking $\alpha_{n-1}+\alpha_n$ to β as $a_{\beta-\alpha_n,n-1}$ fixes α_n. Now $a_{\beta-\alpha_n,n-1}r_{n-1}$ is a shortest word taking α_n to β and so $a_{\beta-\alpha_n,n-1}r_{n-1} \leftrightsquigarrow a_{\beta,n}$. This gives $r_n a_{\beta-\alpha_n,n}e_n \leftrightsquigarrow a_{\beta-\alpha_n,n-1}r_{n-1}e_n \leftrightsquigarrow a_{\beta,n}e_n$.
If $n \notin \mathrm{Supp}\ (\beta)$ let $i = \mathrm{Proj}(\beta, n)$. If $i > 3$, the argument above applies directly with i instead of n and $j \leq i$, giving $r_j a_{\beta-\alpha_j,i} \leftrightsquigarrow a_{\beta,n}e_i$. The assertion now follows from right multiplication by $e_{i+1,n}$. For $i = 3$, the arguments are similar. □

Remark 3.5. *As the proof uses the relation (HNree) which is not binomial in the BMW algebra, two homogeneously equivalent words of (ii) do not necessarily have the same image under ρ in the BMW algebra. Indeed, if $j = n$ and $\beta = \alpha_{n-1} + \alpha_n$, then $a_{\beta,n} = r_{n-1}$ and $a_{\beta-\alpha_n,n} = e_{n-1}$, so $\rho(a_{\beta,n}e_n) = g_{n-1}e_n$ is distinct from $\rho(r_n a_{\beta-\alpha_n,n}e_n) = g_n e_{n-1}e_n$.*

We have denoted elements of F_n by $\underline{a}$. In the remainder of the paper we will need to reduce words which have specific r_i or e_i in them. It is notationally awkward to have long strings underlined, and so we will dispense with this for words including such r_i and e_i. For example we write $\underline{a}r_i r_j e_i \rightsquigarrow \underline{a}e_j e_i$ rather than $\underline{a r_i r_j e_i} \rightsquigarrow \underline{a e_j e_i}$. We continue to underline general elements of F_n as $\underline{a}$.

Let M be a Coxeter diagram with n nodes. The Tits rewrite rules of type M on $s_1,\ldots,s_k$ are the following rewrite rules in the free monoid on $s_1,\ldots,s_n$; cf. [20].

$$\begin{array}{rcl} s_i s_i & \rightsquigarrow & 1 \\ s_i s_j & \rightsquigarrow & s_j s_i \ \text{ if } \ i \not\sim j \\ s_i s_j s_i & \rightsquigarrow & s_j s_i s_j \ \text{ if } \ i \sim j \end{array}$$

Note that the second and the third rule are homogeneous.

Lemma 3.6. *Let M be a Coxeter diagram with n nodes. Then any two reduced words with respect to the Tits rewrite rule of type M on $s_1,\ldots,s_n$ are homogeneously equivalent, that is, can be rewritten into each other by means of a series of the second and the third rewrite rules.*

Proof. The result can be found in [20] and is independently proved in [15]. $\square$

As a first application, note that, for the subgroup W of $\mathbf{BrM}(\mathrm{D}_n)$, the rewrite rules with $r_1, \ldots, r_n$ instead of $s_1, \ldots, s_k$ coincide with (RSrr), (HCrr), and (HNrrr) of Table 2. Therefore, each element of W corresponds to a unique reduced word of F_n up to homogeneous equivalence. In other words, the equivalence classes in F_n of reduced words over $\{r_1, \ldots, r_n\}$ correspond bijectively with the elements of the Coxeter group W. This implies that, for each reduced word $\underline{a} \in F_n$ all of whose symbols are in $\{r_1, \ldots, r_n\}$, its homogeneous equivalence class is uniquely determined by $\pi(\underline{a})$. In Proposition 3.9, we will generalize this application, using words $\underline{s}_i$ in F_n to represent elements of $\mathbf{BrM}(\mathrm{D}_n)$ lying inside the ideal generated by $e_{Y(t)}$.
A slightly less general statement holds for $\mathbf{B}(\mathrm{D}_n)$ instead of $\mathbf{BrM}(\mathrm{D}_n)$. As the above-mentioned rewrite rules are binomial in Table 1 as well, for each reduced $\underline{a} \in F_n$ all of whose symbols are in $\{r_1, \ldots, r_n\}$, its homogeneous equivalence class is uniquely determined by $\rho(\underline{a})$ as well. In Proposition 6.3, we will generalize this application, using the same words $\underline{s}_i$ as above in F_n to represent elements of $\mathbf{B}(\mathrm{D}_n)$ lying inside the ideal generated by $e_{Y(t)}$.
Observe that F_{n-1} is a submonoid of F_n.

Lemma 3.7. *Let $\underline{z}_n^* = e_{n,2}r_1e_{3,n}$ for $n \geq 3$ and $\underline{z}_n^* = r_{2-n}e_n$ for $n \in \{1,2\}$, all of these viewed as words in F_n. Then $\underline{z}_n^*$ has height 1 and satisfies the following reductions for $n \geq 3$.*

(i) $r_n^* e_n \rightsquigarrow \underline{z}_n^*$ *and* $e_n r_n^* \rightsquigarrow \underline{z}_n^*$.
(ii) $\underline{z}_n^* \leftrightsquigarrow e_{n,3}r_2e_1e_{3,n}$.
(iii) *For $n \geq 4$ and $i \in \{1, \ldots, n-2\}$,* $e_i\underline{z}_n^* \leftrightsquigarrow \underline{z}_i^* e_n \leftrightsquigarrow e_n\underline{z}_i^*$ *and* $r_i\underline{z}_n^* \leftrightsquigarrow \underline{z}_n^* r_i$.
(iv) $\underline{z}_n^* e_{n-2} \leftrightsquigarrow e_n\underline{z}_{n-2}^*$ *and* $e_n\underline{z}_n^* \leftrightsquigarrow \underline{z}_n^* e_n \leftrightsquigarrow \delta\underline{z}_n^*$.
(v) $\underline{z}_n^*\underline{z}_n^* \rightsquigarrow \delta e_n$.

Proof. By definition, there is only one factor r_i in $\underline{z}_n^*$ and so its height equals 1.

(i). Let $w_{2,n} = r_3r_2r_4r_3r_5r_4\cdots r_{n-1}r_{n-2}r_nr_{n-1}$ be as in [4, Lemma 3.1] and set $w_{n,2} = w_{2,n}^{\mathrm{op}}$. Then $r_n^* = w_{n,2}r_1w_{2,n}$. In order to show the required reductions, we use repeatedly the reducing relation (RNrre), that is, $r_jr_ie_j \rightsquigarrow e_ie_j$ for $i \sim j$, which holds when i and j differ by 1 (excluding $\{1,2\}$), as the triple node is 3 here. In particular, $w_{n,2}e_n \rightsquigarrow e_{2,n}$. Now $r_1e_{2,n} \rightsquigarrow e_2r_1e_{3,n}$ and $r_n^*e_n = w_{n,2}r_1w_{2,n}e_n \rightsquigarrow e_{n,2}r_1e_{3,n} = \underline{z}_n^*$. A similar computation shows that $e_nr_n^* \rightsquigarrow \underline{z}_n^*$.

(ii). This statement holds because $e_3e_2r_1e_3 \leftrightsquigarrow e_3r_2e_1e_3$ is immediate from the defining relation (HTeere).

(iii). For $i \in \{2, \ldots, n-2\}$, by the definition of $e_{k,n}$, (HCee), and (HNeee),

$$\begin{aligned} e_i\underline{z}_n^* &\leftrightsquigarrow e_{n,i+2}e_ie_{i+1}e_{i,2}r_1e_{3,n} \leftrightsquigarrow e_{n,i+2}e_ie_{i,2}r_1e_{3,n} \\ &\leftrightsquigarrow e_{n,i+2}e_{i,2}r_1e_{3,i}e_{i+1}e_{i+2,n} \leftrightsquigarrow e_{i,2}r_1e_{3,i}e_{n,i+2}e_{i+1}e_{i+2,n} \\ &\leftrightsquigarrow \underline{z}_i^*e_n \leftrightsquigarrow e_n\underline{z}_i^*. \end{aligned}$$

By (HCer), (HNree), and (HNeer),

$$\begin{aligned}
r_i\underline{z}_n^* &\leftrightsquigarrow e_{n,i+2}r_ie_{i+1}e_{i,2}r_1e_{3,n} \leftrightsquigarrow e_{n,i+2}r_{i+1}e_ie_{i-1,2}r_1e_{3,n}\\
&\leftrightsquigarrow e_{n,i+2}e_{i+1}r_{i+2}e_{i,2}r_1e_3e_{4,n} \leftrightsquigarrow e_{n,2}r_1e_{3,i}r_{i+2}e_{i+1}e_{i+2,n}\\
&\leftrightsquigarrow e_{n,2}r_1e_{3,i}r_{i+1}e_{i+2,n} \leftrightsquigarrow e_{n,2}r_1e_{3,i}e_{i+1}r_ie_{i+2,n}\\
&\leftrightsquigarrow \underline{z}_n^*r_i.
\end{aligned}$$

The case $i = 1$ is notationally different but can be done the same way as $i = 2$.

(iv). In view of the palindromic nature of the word $\underline{z}_i^*$ and the fact, proved in (iii), that $\underline{z}_i^*$ and e_n commute homogeneously, we see that e_i and $\underline{z}_n^*$ commute homogeneously. Applying this with $i = n-2$ gives $\underline{z}_{n-2}^*e_n \leftrightsquigarrow e_n\underline{z}_{n-2}^*$. The second chain of homogeneous equivalences is a direct consequence of (RSee).

(v). By (RSee), (HCer), (HNeee), and (RSrr),

$$\begin{aligned}
\underline{z}_3^*\underline{z}_3^* &= e_3e_2r_1e_3e_3e_2r_1e_3 \rightsquigarrow \delta e_3r_1e_2e_3e_2r_1e_3 \rightsquigarrow \delta e_3r_1e_2r_1e_3\\
&\rightsquigarrow \delta e_3r_1r_1e_2e_3 \rightsquigarrow \delta e_3e_2e_3 \rightsquigarrow \delta e_3.
\end{aligned}$$

Also, by (HSee), (HCer), (HNeee), and (RSrr),

$$\begin{aligned}
\underline{z}_n^*\underline{z}_n^* &= e_{n,4}e_3r_1e_2e_3e_{4,n}e_{n,4}e_3r_1e_2e_3e_{4,n} \rightsquigarrow \delta e_{n,4}e_3r_1e_2e_3e_2r_1e_3e_{n,4}\\
&\rightsquigarrow \delta e_{n,4}e_3e_{4,n} \rightsquigarrow \delta e_n.
\end{aligned}$$

The cases $n = 1$ and $n = 2$ can be done separately. $\square$

Notation 3.8. Let $\underline{z}_n^*$ be as in Lemma 3.7, and $t \in \{0,\ldots,\lfloor n/2\rfloor\}$. Write $1_t = e_{Y(t)}\delta^{-t}$. So, if $t = 0$, then $Y(t) = \emptyset$ and $1_0 = 1$. If $t = 1$, then $Y(t) = \{\alpha_n\}$, and $1_1 = e_n\delta^{-1}$. Similarly, write $1'_t = e_{Y'(n/2)}\delta^{-n/2}$ if n is even.

We let $C_{Y(t)}$ denote the diagram $C_{WY(t)}$ but with the nodes labeled such that the single node in the component of type A_1 is labeled 0 (if $n-2t = 2$, there are three A_1-components, but then the nodes are $\{0,1,2\}$ and each node represents a component) and the labels $1,2,\ldots,n-2t$ of the component D_{n-2t} are distributed as in Figure 1.

To appreciate the distinction between $C_{Y(t)}$ and $C_{WY(t)}$, observe that, for $n = 6$ and $t = 2$, the maximal element in the W-orbit of $Y(2) = \{\alpha_4,\alpha_6\}$ is $X = \{\varepsilon_3 + \varepsilon_6, \varepsilon_4 + \varepsilon_5\} = \{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6\}$, so $C_{WY(2)} = \{i \in \{1,\ldots,6\} \mid \alpha_i \in X^\perp\} = \{1,2,5\}$ whereas $C_{Y(2)} = \{0,1,2\}$. More generally, $C_{Y(t)} = \{0,1,2,\ldots,n-2t\}$ and $C_{WY(t)} = \{1,2,\ldots,n-2t,n-t+1\}$.

We distinguish the following elements of F_n.

$$\underline{s}_0 = \underline{z}_n^*\delta^{-1}1_t, \qquad \underline{s}_i = r_i1_t \quad (1 \le i \le n-2t). \tag{2}$$

In the proposition below we establish the Tits rewrite rules for the Coxeter group of type $C_{Y(t)}$ with generators as in (2) and identity 1_t.

Proposition 3.9. *Let $n \ge 4$ and $t \in \{0,\ldots,\lfloor n/2\rfloor\}$. The words $\underline{s}_i$ for i a node of $C_{Y(t)}$ have height 1 and satsify the following properties.*

(i) With respect to the rewrite system of Table 2 in F_n, the words $\underline{s}_i$ for i nodes of $C_{Y(t)}$ of F_n satisfy the Tits rewrite rules of type $C_{Y(t)}$ with identity element 1_t. That is, they satisfy $1_t1_t \leftrightsquigarrow 1_t$, $\underline{s}_i1_t \leftrightsquigarrow \underline{s}_i$, $1_t\underline{s}_i \leftrightsquigarrow \underline{s}_i$, $\underline{s}_i\underline{s}_i \rightsquigarrow 1_t$, $\underline{s}_i\underline{s}_j \leftrightsquigarrow \underline{s}_j\underline{s}_i$ if $i \not\sim j$, and $\underline{s}_i\underline{s}_j\underline{s}_i \leftrightsquigarrow \underline{s}_j\underline{s}_i\underline{s}_j$ if $i \sim j$, where i and j are nodes of $C_{Y(t)}$.

(ii) The elements $\pi(\underline{s}_i)$, for i running through the nodes of $C_{Y(t)}$, generate a Coxeter group of type $C_{Y(t)}$ in $\mathbf{BrM}(\mathrm{D}_n)$ with identity element $\pi(1_t) = e_{Y(t)}\delta^{-t}$.

(iii) Let $U_{Y(t)}$ be the set of words in $F_n 1_t$ that are minimal expressions in the $\underline{s}_i$ (i nodes of $C_{Y(t)}$) for elements of the Coxeter group of (ii). Then the restriction of π to $U_{Y(t)}$ induces a bijection from the set of homogeneous equivalence classes in $U_{Y(t)}$ onto this Coxeter group.

Moreover, similar statements hold for $Y'(n/2)$ instead of $Y(n/2)$ in case n is even.

Proof. Recall $\mathrm{ht}(e_{Y(t)}) = 0$. By Lemma 3.7(ii), $\mathrm{ht}(\underline{z}_n^*) = 1$, so $\mathrm{ht}(\overline{s}_0) = 1$, and, clearly, $\mathrm{ht}(\underline{s}_i) = 1$ for $i > 0$.

(i). We verify the individual rewrite rules. Those involving 1_t at the left hand side are straightforward applications of the rules (HSee), (HCee), (HCre), and (HNeee).

$\underline{s}_i\underline{s}_i \rightsquigarrow 1_t$. By Lemma 3.7(iv) we see $e_n\underline{z}_n^* \leftrightsquigarrow \underline{z}_n^* e_n$ and $\underline{z}_n^* e_j \leftrightsquigarrow e_j\underline{z}_n^*$ for $j \le n-2$ and so $\underline{z}_n^* e_{Y(t)} \leftrightsquigarrow e_{Y(t)}\underline{z}_n^*$. Hence $\underline{z}_n^*$ commutes homogeneously with $e_{Y(t)}$ and so in the definition of $\overline{s}_0$ it does not matter on which side $e_{Y(t)}$ occurs. In particular, using Lemma 3.7(v), we find $\underline{s}_0\underline{s}_0 \leftrightsquigarrow \underline{z}_n^*\underline{z}_n^* e_{Y(t)}e_{Y(t)}\delta^{-2-2t} \rightsquigarrow \delta e_n e_{Y(t)}\delta^{-2-t} \rightsquigarrow e_{Y(t)}\delta^{-t}$, which is the identity element of $\pi(U_{Y(t)})$. This settles the case $i = 0$. For $i > 0$, the assertion $\underline{s}_i\underline{s}_i \rightsquigarrow 1_t$ follows directly from the fact that $e_{Y(t)}$ and r_i commute and (HSrr).

$\underline{s}_i\underline{s}_j \rightsquigarrow \underline{s}_j\underline{s}_i$ if $i \not\sim j$. For $i = 0$ and $j > 0$, this follows from Lemma 3.7(iii). For $i > 0$ and $j > 0$, it is immediate from (HCrr).

$\underline{s}_i\underline{s}_j\underline{s}_i \rightsquigarrow \underline{s}_j\underline{s}_i\underline{s}_j$ if $i \sim j$. Here we must have $i, j > 0$. Now it is immediate from (HNrrr).

(ii). The fact that the $\pi(\underline{s}_i)$ ($0 \le i \le n-2t$) generate a quotient of the Coxeter group of type $C_{Y(t)}$ is immediate from (i) and the fact that a rewrite rule $x \rightsquigarrow y$ in F_n implies $\pi(x) = \pi(y)$. Therefore, it suffices to show that there is a surjective homomorphism from the group U generated by the $\pi(\underline{s}_i)$ onto $W(C_{WY(t)})$. The linear representation $\rho_{WY(t)}$ of [3, Theorem 3.6] gives such a surjective homomorphism. Indeed, in the notation of [loc. cit.], U stabilizes the 1-space in $V_{WY(t)}$ spanned by the vector $\xi_{Y(t)}$ and the group homomorphism $\varphi : U \to W(C_{WY(t)})$ determined by $\rho_{WY(t)}(u)\xi_{Y(t)} = \xi_{Y(t)}\varphi(u)$ is as required. To see this, notice that $\varphi(\pi(\underline{s}_i)) = r_i$ for $i > 0$ and $\varphi(\pi(\underline{s}_0)) = r_{n-t+1}$; for, by [3, (2)], if $i > 0$ we have $\rho_{WY(t)}(\pi(\underline{s}_i))\xi_{Y(t)} = \rho_{WY(t)}(r_i)\xi_{Y(t)} = \xi_{Y(t)}h_{Y(t),i}$, so $\varphi(\pi(\underline{s}_i)) = h_{Y(t),i}$ and a computation using [5, Proposition 4.2] shows $h_{Y(t),i} = r_i$; similarly, $\rho_{WY(t)}(\pi(\underline{s}_0))\xi_{Y(t)} = \xi_{Y(t)}h_{\{\alpha_1\}\cup Y(t-1),2}$ and $h_{\{\alpha_1\}\cup Y(t-1),2} = r_{n-t+1}$. Therefore, the subgroup of W generated by the $\pi(\underline{s}_i)$ must be isomorphic to $W(C_{Y(t)})$.

(iii). This is immediate from (ii) and Lemma 3.6. □

There are some standard properties of the root systems we are using which we mention here for convenience. By our restriction to the simply laced case, all roots have square norm 2. The inner products are all ± 2, ± 1, or 0. If $(\beta, \gamma) = 1$, then $\beta - \gamma$ is a root, and if $(\beta, \gamma) = -1$, then $\beta + \gamma$ is a root. Further if $(\beta, \gamma) = 0$, then $\beta \pm \gamma$ is never a root. We often encounter the situation in which $(\beta, \alpha_i) = 0$, $i \sim j$, and $\beta - \alpha_j$ is a root. Then $(\beta - \alpha_j, \alpha_i) = 0 + 1 = 1$ and so $\beta - \alpha_j - \alpha_i$ is also a root.

The lemma below is a step towards the anticipated counterpart of Theorem 3.2 in terms of reduction. It deals with left multiplication of the element $a_{\beta,n}e_n$, which

is in reduced form, by a generator. In the action of Proposition 2.7, this element maps $\emptyset$ to $\{\beta\}$, so after left multiplication by e_i it will map $\emptyset$ to $\{\alpha_i\}$ or (in case $\alpha_i \perp \beta$) to $\{\alpha_i, \beta\}$, and, after left multiplication with r_i, it will map $\emptyset$ to $\{r_i\beta\}$. The lemma will find corresponding reduced words. In order to control the kernel of this action, we need a little more notation.

Notation 3.10. For $0 \leq t \leq n/2$, let $Z_{Y(t)}$ be the subsemigroup of F_n generated by $1_t\delta^i$ ($i \in \mathbb{Z}$), $\underline{s}_0$, $\underline{s}_j = r_j 1_t$ as in (2), and $e_j 1_t$ for j running through the nodes of $C_{Y(t)}$ with $j > 0$.
If n is even, then $Z_{Y'(n/2)}$ is defined as the subsemigroup of F_n generated by $1_t'$.
We also write Z_n instead of $Z_{Y(1)}$.

For n odd and $t = (n-1)/2$, this means that $Z_{Y(t)}$ is the subsemigroup of F_n generated by $1_t\delta^i$ ($i \in \mathbb{Z}$) and $\underline{s}_0$. For $t = 0$, the subsemigroup $Z_{Y(t)}$ coincides with F_n. If n is even and $t = n/2$, then $Z_{Y(t)}$ is nothing but the subsemigroup of F_n generated by 1_t, which parallels the definition of $Z_{Y'(n/2)}$.

Lemma 3.11. *Let $M = \mathrm{D}_n$. For $i \in \{1, \ldots, n\}$ and β a positive root, the word $e_i a_{\beta,n} e_n$ either belongs to $\pi^{-1}(\Theta)$ or can be reduced to a word in $a_{\beta',n} Z_n$ where β' is a positive root with $\mathrm{ht}(\beta') \leq \mathrm{ht}(\beta)$, and $r_i a_{\beta,n} e_n$ can be reduced to a word in $a_{r_i\beta,n} Z_n$. Moreover, if $\underline{a} \in F_n$ and $\underline{a} e_n$ is not in $\pi^{-1}(\Theta)$, then $\underline{a} e_n$ can be reduced to a word in $a_{\beta',n} Z_n$, where β' is a positive root with $\mathrm{ht}(\beta') \leq \mathrm{ht}(\underline{a})$.*

Proof. We proceed by induction on $\mathrm{ht}(\beta)$. If $\mathrm{ht}(\beta) = 1$ we have $\beta = \alpha_j$ for some node j of D_n. By Lemma 3.4, $a_{\beta,n} e_n \rightsquigarrow e_{j,n}$.
Consider first $e_i a_{\beta,n} e_n$. By the above, $e_i a_{\beta,n} e_n \rightsquigarrow e_i e_{j,n}$. If $\{i,j\} = \{1,2\}$, then $e_i e_{j,n}$ is in $\pi^{-1}(\Theta)$. By symmetry of the diagram, the case $j = 1$ can be replaced by $j = 2$ and handled in a similar way, so assume $j \geq 2$. If $i < j$, then e_i can be commuted to the right and be absorbed into Z_n. If $i = j-1$, then we may assume $i \geq 2$ as we already handled the case $\{i,j\} = \{1,2\}$, and so $e_i a_{\beta,n} e_n \rightsquigarrow e_i e_{j,n} = e_{i,n}$. If $i = j$ we obtain $e_i a_{\beta,n} e_n \rightsquigarrow \delta e_{i,n}$. If $i = j+1$ we can use $e_i e_j e_i \leftrightsquigarrow e_i$ to get $e_i a_{\beta,n} e_n \rightsquigarrow e_{i,n}$. Otherwise commute the e_i past terms in $e_{j,n}$ to obtain $e_i a_{\beta,n} e_n \rightsquigarrow e_j e_{j+1} \cdots e_i e_{i-1} e_i \cdots e_n$. Now use $e_i e_{i-1} e_i \leftrightsquigarrow e_i$ and commute the preceding terms to the right and absorb it in Z_n. In each of these cases $e_i a_{\beta,n} e_n \rightsquigarrow a_{\beta',n} z$ for some $z \in Z_n$ and some $\beta' \in \{\alpha_i, \alpha_j\}$, as required.
We now consider $r_i a_{\beta,n} e_n$ with $\beta = \alpha_j$ for some node j and $r_i a_{\beta,n} e_n \rightsquigarrow r_i e_{j,n}$. There are two special cases which we handle directly, viz., $j = 2$ with $i = 1$ and $j = 1$ with $i = 2$. For the first we have $r_1 e_{2,n} \rightsquigarrow e_2 r_1 e_{2,n} \delta^{-1}$. Consequently, $e_2 \rightsquigarrow e_{2,n} e_{n,2} \delta^{-1}$ and so $r_1 e_{2,n} \rightsquigarrow e_2 r_1 e_{3,n} \rightsquigarrow e_{2,n} e_{n,2} r_1 e_{3,n} \delta^{-1} = e_{2,n} \underline{z}_n^* \delta^{-1} = a_{\alpha_2,n} \underline{z}_n^* \delta^{-1}$ and we are as $\underline{z}_n^* \delta^{-1}$ belongs to Z_n. The other case is similar. Assume, therefore, that these special cases do not occur. If $i = j-1$, we have $r_i e_{j,n-1} e_n = a_{r_i\beta} e_n$ and we are done. If $i < j-1$, then r_i commutes homogeneously through to give $e_{j,n} r_i$, unless we have $i = 1$ and $j = 3$, a case that can be treated as $i = 2$ and $j = 3$, which is done below; observe that the expression $e_{j,n} r_i$ satisfies all the conditions needed. If $i = j$ use $r_i e_i \rightsquigarrow e_i$ to see that $r_i e_{j,n-1} e_n \rightsquigarrow e_{i,n} \in a_{\beta,n} Z_n$. As above if $i = j+1$, then by (HNree), $r_{j+1} e_j e_{j+1} e_{j+2,n} \rightsquigarrow r_j e_{j+1} e_{j+2,n} = r_j e_{j+1,n} = a_{\alpha_j + \alpha_{j+1},n} e_n$. This is what is required as here $\beta = \alpha_j$ and $r_{j+1}\alpha_j = \alpha_j + \alpha_{j+1}$. Otherwise, $i > j+1$ and $r_i e_{j,n} \rightsquigarrow e_{j,i-2} r_i e_{i-1} e_{i,n} \rightsquigarrow e_{j,i-2} r_{i-1} e_{i,n}$. Now $e_{j,i-2} = e_{j,i-3} e_{i-2}$, and use $e_{i-2} r_{i-1} \rightsquigarrow e_{i-2} e_{i-1} r_{i-2}$ to get $e_{j,i-1} r_{i-2} e_{i,n}$ and commute r_{i-2} homogeneously to the right. This gives the required form.

We may suppose then that β has height greater than 1 and so there is a node j for which $\beta - \alpha_j$ is a root. Throughout this part of the proof we use Lemma 3.4 when $\beta - \alpha_j$ is a root to see that up to homogeneous equivalence $a_{\beta,n}e_n$ and $r_j a_{\beta-\alpha_j}e_n$ are the same.

Again, consider first $e_i a_{\beta,n}e_n$. Choose $j = i$ if possible. If so, we use $e_i r_i \rightsquigarrow e_i$ to obtain $e_i r_i a_{\beta-\alpha_i,n}e_n \rightsquigarrow e_i a_{\beta-\alpha_i,n}e_n$. The resulting word has lower height than $e_i r_i a_{\beta-\alpha_i,n}e_n$ and we use induction to finish. Suppose $i \not\sim j$ and $i \neq j$. Then $e_i r_j a_{\beta-\alpha_j,n}e_n \rightsquigarrow r_j e_i a_{\beta-\alpha_j,n}e_n$. Now apply the induction hypothesis to $e_i a_{\beta-\alpha_j,n}e_n$ so $e_i a_{\beta-\alpha_j,n}e_n \rightsquigarrow a_{\beta',n}e_n z$ where $z \in Z_n$ and $\mathrm{ht}(\beta') < \mathrm{ht}(\beta)$. In view of this inequality, induction applies to the statement Involving $r_j a_{\beta',n}e_n$. Acting by r_j could raise the height at most one, still leaving $\mathrm{ht}(r_j\beta') \leq \mathrm{ht}(\beta)$ as needed. Suppose $i \sim j$. We know that (α_i, β) is not 1 as we have chosen $j = i$ if possible above. This means either $(\beta, \alpha_i) = 0$ or $(\beta, \alpha_i) = -1$. Suppose first $(\beta, \alpha_i) = 0$. Then $(\beta - \alpha_j, \alpha_i) = 1$ and so $\beta - \alpha_j - \alpha_i$ is a root and $a_{\beta,n}e_n = r_j r_i a_{\beta-a_j-\alpha_i,n}e_n$; now $e_i r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n \rightsquigarrow e_i e_j a_{\beta-\alpha_j-\alpha_i,n}e_n$, and we can finish by induction to get the result as the height of the root $\beta - \alpha_j - \alpha_i$ is at most $\mathrm{ht}(\beta) - 2$. Suppose now $(\beta, \alpha_i) = -1$. Then $e_i a_{\beta,n}e_n \rightsquigarrow e_i r_j a_{\beta-\alpha_j,n}e_n \rightsquigarrow e_i e_j r_i a_{\beta-\alpha_j,n}e_n$. Now notice $(\beta - \alpha_j, \alpha_i) = -1 + 1 = 0$ and so $r_i(\beta - \alpha_j) = \beta - \alpha_j$ and so $e_i a_{\beta,n}e_n \rightsquigarrow e_i e_j r_i a_{\beta-\alpha_j,n}e_n \rightsquigarrow e_i e_j a_{\beta-\alpha_j,n}e_n z$ by the induction hypothesis for the action of r_i. Using the induction hypothesis twice more, we find $e_i a_{\beta,n}e_n \rightsquigarrow e_i a_{\beta',n}e_n z' \rightsquigarrow a_{\beta''}e_n z''$ for certain roots β' and β'' whose height is at most $\mathrm{ht}(\beta) - 1$. This ends the part of the proof involving left multiplication by e_i.

We now consider $r_i a_{\beta,n}e_n$ with $\mathrm{ht}(\beta) > 1$ and $\beta - \alpha_j$ is a root for some node j. If $(\beta, \alpha_i) = -1$, then $r_j a_{\beta,n}e_n \leftrightsquigarrow a_{\beta+\alpha_i,n}e_n$ by Lemma 3.4 and we are done. Suppose $(\beta, \alpha_i) = 1$. Then $\beta - \alpha_i$ is a root and $a_{\beta,n}e_n \leftrightsquigarrow r_i a_{\beta-\alpha_i,n}e_n$. Now use $r_i r_i a_{\beta-\alpha_i,n}e_n \rightsquigarrow a_{\beta-\alpha_i,n}e_n \leftrightsquigarrow a_{r_i\beta,n}e_n$ to finish.

Therefore, we can assume $(\beta, \alpha_i) = 0$. There is a node j for which $\beta - \alpha_j$ is a root and so $r_i a_{\beta,n}e_n \leftrightsquigarrow r_i r_j a_{\beta-\alpha_j,n}e_n$ by Lemma 3.4. The arguments here are similar to the ones at the beginning of this proof when $\mathrm{ht}\,\beta > 1$. In particular, if $i \not\sim j$ and $i \neq j$ this is $r_j r_i a_{\beta-\alpha_j,n}e_n$ and use induction for r_i acting in the case $(\alpha_i, \beta - \alpha_j) = 0$.

The only remaining case is $i \sim j$ and still $(\beta, \alpha_i) = 0$. Here $\beta - \alpha_i - \alpha_j$ is a root orthogonal to α_j and $a_{\beta,n}e_n \leftrightsquigarrow r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n$ by Lemma 3.4. We consider $r_i r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n$ and so use the homogeneous relation $r_i r_j r_i \leftrightsquigarrow r_j r_i r_j$, the induction hypothesis and $(\alpha_j, \beta - \alpha_j - \alpha_i) = 0$ to reduce $r_i r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n \rightsquigarrow r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n z \rightsquigarrow a_{\beta,n}e_n z'$ with $z, z' \in Z_n$ as required. This proves the first part of the lemma.

As for the second statement, without loss of generality, we may assume that $\underline{a}e_n$ is reduced and does not belong to $\pi^{-1}(\Theta)$. We argue by induction on the length of $\underline{a}$. Whenever $\underline{a}$ is equal to $a_{\beta,n}$, there is nothing to show. In particular, we may assume that $\underline{a}$ has positive length; say it starts with e_i or r_i. By induction, we have $\underline{a}e_n \rightsquigarrow e_i a_{\beta,n}e_n z$ or $\underline{a}e_n \rightsquigarrow r_i a_{\beta,n}e_n z$ with $\mathrm{ht}(\beta) \leq \mathrm{ht}(\underline{a})$ for some $z \in Z_n$. The proof now follows from the first statement. $\square$

Lemma 3.12. *Suppose that $\beta \in \Phi$ and $w \in W$ satisfy $w\alpha_n = \beta$. Then each reduction of we_n is homogeneously equivalent to a word in $a_{\beta,n}Z_n$.*

Proof. Let $\underline{a}$ be a reduced word with $we_n \rightsquigarrow \underline{a}$. By the definition of the Brauer monoid action in Proposition 2.7, $e_n(\emptyset) = \{\alpha_n\}$ and $we_n(\emptyset) = \{w\alpha_n\} = \{\beta\}$. In general, if $\underline{a}$, $\underline{b}$ in F_n satisfy $\underline{a} \rightsquigarrow \underline{b}$, then $\pi(\underline{a})(B) = \pi(\underline{b})(B)$ for any admissible set B. In particular, we must have $\pi(\underline{a})(\emptyset) = \{\beta\}$.
Now $\pi((we_n)^{\mathrm{op}})(\emptyset) = \{\alpha_n\}$ as $w^{\mathrm{op}}(\emptyset) = \emptyset$ and $e_n(\emptyset) = \{\alpha_n\}$. In turn we see $\pi(\underline{a}^{\mathrm{op}})(\emptyset) = \{\alpha_n\}$. This means there must be an occurrence of e_i in $\underline{a}$ for some i as otherwise $\pi(\underline{a}^{\mathrm{op}})(\emptyset)$ would be $\emptyset$. If $i \neq n$, rewrite $\underline{a}$ using $e_i \leftrightsquigarrow e_{i,n}e_{n-1,i}$ to create an occurrence of e_n in $\underline{a}$. Write $\underline{a} = \underline{b}e_n\underline{c}$, with $\underline{b}$ and $\underline{c}$ in F_n such that e_n does not occur in $\underline{b}$. Observe that $\pi(\underline{a}) = we_n \notin \pi(\Theta)$, so, by Lemma 3.11 applied to $\underline{b}e_n$ and $\underline{c}^{\mathrm{op}}e_n$, we can reduce $\underline{a}$ to $a_{\beta',n}za^{\mathrm{op}}_{\alpha',n}$ for some $z \in Z_n$. Because of the definition of Z_n we can write $z = e_nz'$ where z' is also in Z_n.
By the definition of the action in Proposition 2.7, $e_{k,n}(\emptyset) = \{\alpha_k\}$ and, $e_na^{\mathrm{op}}_{\alpha',n}(\emptyset) = \{\alpha_n\}$ as $e_n\{\alpha_{n-1}\} = \{\alpha_n\}$ and $e_n(\emptyset) = \{\alpha_n\}$. Moreover, $\pi(z')\{\alpha_n\} = \{\alpha_n\}$ as $z \in Z_n$, which, according to Notation 3.10, is generated by elements r_i for $i \leq n-2$ fixing $\{\alpha_n\}$ and e_i for $i \leq n-2$ whose action on $\{\alpha_n\}$ would add an α_i to $\{\alpha_n\}$ and so do not occur in z. Similarly, $\pi(\underline{z}^*_n)$ fixes $\{\alpha_n\}$. Now $\{\beta\} = w\{\alpha_n\} = we_n(\emptyset) = a_{\beta',n}\pi(z')e_na^{\mathrm{op}}_{\alpha',n}(\emptyset) = a_{\beta',n}z'\{\alpha_n\} = a_{\beta',n}\{\alpha_n\} = \{\beta'\}$, so $\beta' = \beta$. Similarly, $\{\alpha_n\} = (we_n)^{\mathrm{op}}(\emptyset) = \{\alpha'\}$, so $\alpha' = \alpha_n$. Therefore, $a_{\beta',n}za^{\mathrm{op}}_{\alpha',n} = a_{\beta,n}za^{\mathrm{op}}_{\alpha_n,n} = a_{\beta,n}z$ as here $a_{\alpha_n,n}$ is the identity. This establishes $\underline{a} \rightsquigarrow a_{\beta,n}z$ for some $z \in Z_n$.
Finally, $\rightsquigarrow$ for reduced expressions is the same as $\leftrightsquigarrow$ and so the statement about homogeneous reduction is sastisfied. □

We now return to the set $Y(t)$ and use $Z_{Y(t)}$ of Notation 3.10. Again $e_{Y(t)}$ commutes homogeneously with the elements $\underline{z}^*_n$ and r_i, e_i ($i = 1, \ldots, n-2t$), so, up to homogeneous equivalence, it does not matter on which side $e_{Y(t)}$ is located in these expressions for elements of $Z_{Y(t)}$.

Lemma 3.13. *For each $t \in \{1, \ldots, \lfloor n/2 \rfloor\}$ and each $\underline{a} \in F_n$ such that $\pi(\underline{a}e_{Y(t)}) \notin \Theta$, the word $\underline{a}e_{Y(t)}$ can be reduced to an element of the form*

$$a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}z \tag{3}$$

with $\beta_{n-2k} \in \Phi^+$ for $0 \leq k \leq t-1$ such that β_{n-2k} has support in D_{n-2k} for each k, and $z \in Z_{Y(t)}$.

Proof. Notice that $a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{Y(t)}$ is homogeneously equivalent to $a_{\beta_n,n}e_na_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}$. By Lemma 3.11, $\underline{a}e_n$ can be reduced to $a_{\beta_n,n}e_n\underline{z}_n$ for some $\beta_n \in \Phi^+$ and $\underline{z}_n \in Z_n$. In particular, up to homogeneous equivalence, cf. Lemma 3.7(iii), we may assume $\underline{z}_n = \underline{a}'$ or $\underline{z}_n = \underline{z}^*_n\underline{a}'$ for some $\underline{a}' \in F_{n-2}$. If $t = 1$, we are done by Lemma 3.11. Otherwise, by induction on n, we find

$$\underline{a}'e_{Y(t)\setminus\{n\}} = a_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}\underline{z}_{n-2t+2}$$

for some $\underline{z}_{n-2t+2} \in Z_{Y(t)\setminus\{e_n\}}$. As $\underline{a} \in F_{n-2}$, we have $\pi(\underline{a}'e_{Y(t)\setminus\{n\}})$, and so the support of β_{n-2} is in D_{n-2}.

For $i \in \{0,1\}$ we will write $(\underline{z}_n^*)^i$ to denote either the identity or $\underline{z}_n^*$ depending on whether i is 0 or 1. Now, by Lemma 3.7, for $i=0$ or $i=1$,

$$\begin{aligned}
\underline{a}e_{Y(t)} &\rightsquigarrow a_{\beta_n,n}e_n(\underline{z}_n^*)^i\underline{a}'e_{Y(t)\setminus\{n\}}\\
&\rightsquigarrow a_{\beta_n,n}e_n(\underline{z}_n^*)^i a_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}\underline{z}_{n-2t+2}\\
&\rightsquigarrow a_{\beta_n,n}e_n a_{\beta_{n-2},n-2}e_{n-2}(\underline{z}_{n-2}^*)^i\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}\underline{z}_{n-2t+2}\\
&\rightsquigarrow a_{\beta_n,n}e_n a_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}(\underline{z}_{n-2t+2}^*)^i\underline{z}_{n-2t+2}\\
&\rightsquigarrow a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}z
\end{aligned}$$

with $z = e_n e_{n-2}\cdots e_{n-2t+2}(\underline{z}_{n-2t+2}^*)^i\underline{z}_{n-2t+2} \in Z_{Y(t)}$ and $\beta_{n-2k}\in\{1,\ldots,n-2k\}$ for each k, as required.
Notice that $\underline{z}_{n-2t+2}^*$ commutes with elements of Z_{n-2t+2} by Lemma 3.7, (RSer) and (RSre), and the fact that $\underline{z}_{n-2t+2}^*$ starts and ends with e_{n-2t+2}. $\square$

We will consider the different ways to write $\underline{a}e_{Y(t)}$ in this form. The case of just two roots will suffice to argue the general case so for the time being we assume $t=2$ and so $Y(t)=\{\alpha_n,\alpha_{n-2}\}$. We consider words of the form $a_{\beta_n,n}a_{\beta_{n-2},n-2}z$ where z is in $Z_{Y(2)}$. We will need a lemma that involves words in the Weyl group that map $\{\alpha_n,\alpha_{n-2}\}$ to $\{\beta,\gamma\}$ in $\mathcal{A}$ and the ways to reduce them.

Lemma 3.14. *Suppose that $\underline{a}\in F_n$ satisfies $\pi(\underline{a})\{\alpha_n,\alpha_{n-2}\}=\{\beta,\gamma\}$. Then of the two possible reductions of $\underline{a}e_{Y(2)}$ as in Lemma 3.13, at least one can be reduced to the other, that is, for some $z\in Z_{Y(2)}$ we have*

$$\begin{aligned}
&\textit{either} && a_{\beta,n}a_{\beta_{n-2},n-2}e_ne_{n-2}\rightsquigarrow a_{\gamma,n}a_{\gamma_{n-2},n-2}e_ne_{n-2}z\\
&\textit{or} && a_{\gamma,n}a_{\gamma_{n-2},n-2}e_ne_{n-2}\rightsquigarrow a_{\beta,n}a_{\beta_{n-2},n-2}e_ne_{n-2}z.
\end{aligned}$$

Proof. Suppose first that either β or γ has n in its support. Without loss of generality, we assume $n\in\mathrm{Supp}(\beta)$. Then $\pi(a_{\beta,n})\in W$ and, by (HNeee),

$$\begin{aligned}
a_{\beta,n}e_na_{\beta_{n-2},n-2}e_{n-2} &\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n-2}e_{n-2}e_n\\
&\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n-2}e_{n-2}e_{n-1}e_{n-2}e_n\\
&\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n}e_{n-2}e_n\\
&\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n}e_ne_{n-2}
\end{aligned}$$

Notice here $\pi(a_{\beta,n})\{\beta_{n-2}\}=\{\gamma\}$ as in the paragraph above. Now Lemma 3.13 gives

$$a_{\beta,n}a_{\beta_{n-2},n}e_ne_{n-2}\rightsquigarrow a_{\gamma,n}e_na_{\gamma_{n-2},n-2}e_{n-2}z$$

for some $z\in Z_{Y(2)}$. In particular the lemma holds in this case.
The only case left is where n is in the support of neither β nor γ. Here we argue by induction on n. The two reductions of $\underline{a}e_ne_{n-2}$ are $a_{\beta,n-1}e_{n-1}e_na_{\beta_{n-2},n-3}e_{n-3}e_{n-2}$ and $a_{\gamma,n-1}e_{n-1}e_na_{\gamma_{n-2},n-3}e_{n-3}e_{n-2}$ up to right multiples by elements of $Z_{Y(2)}$. The occurrences of e_{n-1} and e_{n-3} are due to assumption on the supports of β and γ. Now both $a_{\beta,n-1}e_{n-1}a_{\beta_{n-2},n-3}e_{n-3}$ and $a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}$ belong to F_{n-1}. By induction on n, one can be reduced to the other—up to a right factor from $Z_{\{\alpha_{n-1},\alpha_{n-3}\}}$, say $a_{\beta,n-1}e_{n-1}a_{\beta_{n-2},n-3}e_{n-3}\rightsquigarrow a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}z'$ for $z'\in Z_{\{\alpha_{n-1},\alpha_{n-3}\}}$. Due to (HNeee) and the definition of Z_{n-1} we have $\underline{z}_{n-1}^*e_{n-2}\leftrightsquigarrow e_{n-1}\underline{z}_{n-2}^*$. Terms in z' generated by e_i or r_i with $i<n-3$ are in $Z_{Y(2)}$. If there is $\underline{z}_{n-3}^*$, then, using Lemma 3.7(iv), we can replace it with $\underline{z}_{n-1}^*$. Now for z' generated by elements with index less than $n-4$ we have $z'e_ne_{n-2}=e_ne_{n-2}z'$ with $z'\in Z_{Y(2)}$.

In case $z' = \underline{z}^*_{n-1}$, we find $\underline{z}^*_{n-1}e_ne_{n-2} = \underline{z}^*_{n-1}e_{n-2}e_n = e_{n-1}\underline{z}^*_{n-2}e_{n-2}e_n\delta^{-1} = e_{n-1}e_{n-2}e_n\delta^{-1}\underline{z}^*_{n-2}$, as $e_{n-2} = e^2_{n-2}\delta^{-1}$.
Now multiplication by e_ne_{n-2} of both sides of the reduction

$$a_{\beta,n-1}e_{n-1}a_{\beta_{n-2},n-3}e_{n-3} \quad \rightsquigarrow \quad a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}z'$$

for $z' \in Z_{\{\alpha_{n-1},\alpha_{n-3}\}}$, as in the paragraph above, and application of Lemma 3.7(iv) gives

$$\begin{aligned} a_{\beta,n}e_na_{\beta_{n-2},n-2}e_{n-2} \quad &\rightsquigarrow \quad a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}z'e_{n-2}e_n \\ &\rightsquigarrow \quad a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}e_{n-2}e_nz \\ &\rightsquigarrow \quad a_{\gamma,n}a_{\gamma_{n-2},n-2}z \end{aligned}$$

as required. Here the z is the same as z' unless z' contains $\underline{z}^*_{n-1}$ or $\underline{z}^*_{n-3}$. Here in the case with $\underline{z}^*_{n-1}$ occurring the extra e_{n-1} commutes to the left and when multiplied with the e_{n-1} creates a δ which when multiplied by the δ^{-1} becomes 1. The z is then $\underline{z}^*_{n-2}$. □

This case for just two roots extends to arbitrary size t, as follows.

Lemma 3.15. *Let $t \in \{1, \ldots, \lfloor n/2\rfloor\}$. If $\underline{a}$ is a word in F_n with $\pi(\underline{a})e_{Y(t)} \notin \Theta$, then there are positive roots β_{n-2k} for $k = 0, \ldots, t-1$ such that β_{n-2k} has support in D_{n-2k} for each k and $\underline{a}e_{Y(t)}$ can be reduced to an element of $\underline{b}Z_{Y(t)}$ where*

$$\underline{b} = a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}$$

and every word reduced from $\underline{a}$ as in Lemma 3.13 can also be reduced to a word in $\underline{b}Z_{Y(t)}$.

Proof. Set $B = \pi(\underline{a})Y(t)$. By Lemma 3.13 there is a unique reduction up to right multiplication by elements of $Z_{Y(t)}$ for each ordering of the elements of B. We use Lemma 3.14 to see that the order of, say the first two, does not matter, in the sense that one reduction can be reduced to another. Continuing this way with α_{n-2} and α_{n-4}, we see that the words as in Lemma 3.13 for all orders of the roots of B can be reduced to a particular one. This proves the lemma. □

Notation 3.16. The lemma allows us to define $A_{B,n}$, for $B \in WY(t)$, as the unique element $\underline{b} \in F_n$ up to homogeneous equivalence determined by Lemma 3.15 with $B = \pi(\underline{b})Y(t)$. When $t = 0$, we take $A_{B,n}$ to be the identity.

Theorem 3.17. *Suppose $\underline{a} \in F_n$ satisfies $\pi(\underline{a}) \notin \Theta$. Then, up to an interchange of the nodes 1 and 2 of D_n when $n = 2t$, the word $\underline{a}$ can be reduced to a word of the form $A_{B,n}zA^{\mathrm{op}}_{B',n}\delta^k$ where B, $B' \in WY(t)$ and $z \in U_{Y(t)}$ for $t = |B| \in \{0, \ldots, \lfloor n/2\rfloor\}$.*

Proof. Put $B = \pi(\underline{a})(\emptyset)$ and $B' = \pi(\underline{a}^{\mathrm{op}})(\emptyset)$. It follows from Lemma 3.13 that the two sets belong to the same W-orbit inside $\mathcal{A}$, namely the one containing $Y(t)$. Take $t = |B|$. It suffices to prove the statement of the theorem for $Z_{Y(t)}$ instead of $U_{Y(t)}$ because $\pi(\underline{a})(\emptyset) = B$ and, by Proposition 2.7, the presence of $e_{Y(t)}e_i$ in z for some $i < n-2t$ would imply that $\pi(A_{B,n}zA^{\mathrm{op}}_{B',n})(\emptyset)$ contains $\pi(A_{B,n})(Y(t) \cup \{\alpha_i\})$, a set of size greater than t; however $\pi(\underline{a})(\emptyset) = B$ has size $|B| = t$ and so this is a contradiction.

Suppose $B = \emptyset$. Then $\underline{a}$ must not contain any occurrences of e_i as $e_i(\emptyset)$ contains α_i (cf. the last assertion of Proposition 2.7). This means that $\underline{a}$ is a word for W and the Tits rewrite rules for W suffice for the validity of the theorem in this case, with $t = 0$, $Y(0) = \emptyset$, and $V_\emptyset = W$.
Therefore, we may assume that $B \neq \emptyset$, so there is an index i such that e_i occurs in $\underline{a}$. If $i \neq n$, then by homogeneous equivalence, we can replace e_i by $e_{i,n}e_{n-1,i}$. Thus $\underline{a} = \underline{b}e_n\underline{c}$ for certain $\underline{b}$, $\underline{c} \in F_n$. By Lemma 3.12 applied to both $\underline{b}$ and $\underline{c}^{\mathrm{op}}$, we can reduce $\underline{a}$ to $a_{\beta,n}\underline{z}_n a^{\mathrm{op}}_{\beta',n}$ for some $\beta \in B$, $\beta' \in B'$ and $\underline{z}_n \in Z_n$. Then, by an argument as in the proof of Lemma 3.13, $\underline{z}_n = e_n\underline{a}'(\underline{z}_n^*)^i$ with $\underline{a}' \in F_{n-2}$ and $i \in \{0,1\}$. This deals with the case where $t = 1$.
Suppose $t > 1$. By induction on n, the word $\underline{a}'$ reduces to $A_{D,n-2}z'A^{\mathrm{op}}_{D',n-2}$ for some $z' \in Z_{Y''}$, where $Y'' = \{\alpha_{n-2}, \ldots, \alpha_{n-2t+2}\}$. Due to Lemma 3.7 (ii), (iv),

$$e_ne_{n-2}\underline{z}_n^* \leftrightsquigarrow e_ne_n\underline{z}_{n-2}^* \leftrightsquigarrow \delta e_n\underline{z}_{n-2}^* \leftrightsquigarrow e_ne_{n-2}\underline{z}_{n-2}^*.$$

By induction on t, this gives $e_{Y(t)}\underline{z}_n^* \leftrightsquigarrow e_{Y(t)}\underline{z}_{n-2t+2}^*$, which is the same as $e_ne_{Y''}\underline{z}_n^* \leftrightsquigarrow e_ne_{Y''}\underline{z}_{n-2t+2}^*$. So, by Lemmas 3.15 and 3.7 parts (iii) and (iv),

$$\begin{aligned}
\underline{a} &\rightsquigarrow a_{\beta,n}\underline{z}_n a^{\mathrm{op}}_{\beta',n} \leftrightsquigarrow a_{\beta,n}e_n\underline{a}'(\underline{z}_n^*)^i a^{\mathrm{op}}_{\beta',n} \rightsquigarrow a_{\beta,n}e_nA_{D,n-2}z'A^{\mathrm{op}}_{D',n-2}(\underline{z}_n^*)^i a^{\mathrm{op}}_{\beta',n}\\
&\rightsquigarrow a_{\beta,n}A_{D,n-2}e_nz'(\underline{z}_n^*)^iA^{\mathrm{op}}_{D',n-2}a^{\mathrm{op}}_{\beta',n}\\
&\rightsquigarrow a_{\beta,n}A_{D,n-2}e_nz'(\underline{z}_{n-2t+2}^*)^iA^{\mathrm{op}}_{D',n-2}a^{\mathrm{op}}_{\beta',n} \rightsquigarrow A_{B,n}zA^{\mathrm{op}}_{B',n}
\end{aligned}$$

for some $z \in Z_{Y(t)}$. □

The following corollary extends Lemma 3.6 to the part of the Brauer monoid not in Θ. The words $A_{B,n}$ are introduced in Notation 3.16 and $U_{Y(t)}$ in Proposition 3.9(iii).

Corollary 3.18. *For each $\underline{a} \in F_n$ such that $\pi(\underline{a}) \notin \Theta$, all reduced elements of F_n reducible from $\underline{a}$ are homogeneously equivalent to an element of the form $A_{B,n}zA^{\mathrm{op}}_{B',n}\delta^k$ with B and B' in $WY(t)$ for some $t \in \{0, 1, \ldots, \lfloor n/2\rfloor\}$ (up to interchange of the nodes 1 and 2 if $n = 2t$), and $z \in U_{Y(t)}$. Here the elements B and B' are uniquely determined by $B = \pi(\underline{a})(\emptyset)$ and $B' = \pi(\underline{a}^{\mathrm{op}})(\emptyset)$, respectively.*

Proof. The form is immediate from the theorem. Uniqueness up to homogeneous equivalence follows from Lemma 3.15 for $A_{B,n}$ and $A_{B',n}$ and from Tits' rewrite rules for $z \in U_{Y(t)}$, as stated in Proposition 3.9. □

Now let T be the set of elements $A_{B,n}zA^{\mathrm{op}}_{B',n}$ in F_n as in Corollary 3.18. Then the elements of T correspond to triples (B, B', z) where B and B' are W-images of $Y(t)$ and $z \in U_{Y(t)}$; see [3] Corollary 5.4 and Remark 5.7. In particular, T is a finite set and every $\underline{a} \in F_n$ for which $\pi(\underline{a})$ is not in Θ reduces to an element of T up to a power of δ. As powers of δ are in R, the set T will suffice as a spanning set once it is joined with a spanning set for Θ.

Remark 3.19. The proof of Corollary 3.18 has implications for the ordinary Brauer algebra of type A_{n-1} which we can take to be generated by r_i, e_i for $2 \leq i \leq n$. Here there are no r_i^* and no Θ, and $U_{Y(t)}$ consists of the reduced words on $\underline{s}_i$ for $2 \leq i \leq n-2t$, while W is generated by the r_i for $2 \leq i \leq n$ and is isomorphic to the symmetric group of n points.

4. Reductions for R

In this section, we discuss properties of R which show how to relate some properties of sets of monomials in $\mathbf{B}(\mathrm{D}_n)$ to corresponding ones in $\mathbf{Br}(\mathrm{D}_n)$ using the maps π and ρ. The proof of the main theorem will invoke Proposition 4.3.

Lemma 4.1. *The ring R embeds in $\mathbb{Q}(\delta)[l^{\pm 1}]$ and also in $\mathbb{Q}(l,\delta)$.*

Proof. Let $D = \mathbb{Z}[l^{\pm 1}, \delta^{\pm 1}]$, which is a unique factorization domain, and let F be its field of fractions. Put $s(m) = (1-\delta)m - (l - l^{-1})$. Notice $s(m)$ is primitive and so irreducible in $F(m)$ by Gauss' Lemma. Hence $R = D[m]/(s(m))$ is an integral domain and so its field of fractions is $\mathbb{Q}(l,\delta)$. Finally, $\mathbb{Q}(\delta)[l^{\pm 1}]$ is a subring of $\mathbb{Q}(l,\delta)$ containing D and $m = (l-l^{-1})/(1-\delta)$ modulo $s(m)$, and so also contains R. $\square$

Here is a lemma which will give a lower bound for the rank of $\mathbf{B}(\mathrm{D}_n)$.

Lemma 4.2. *Suppose that T is a finite set of monomials in F_n whose images $\pi(T) = \{\pi(t) \mid t \in T\}$ are linearly independent in $\mathbf{Br}(\mathrm{D}_n)$. Then $\rho(T)$ is a linearly independent set in $\mathbf{B}(\mathrm{D}_n)$.*

Proof. Suppose that $\sum_{t\in T} \lambda_t \rho(t)$ with $\lambda_t \in R$ is a non-trivial linear combination that is equal to 0 in $\mathbf{B}(\mathrm{D}_n)$. This gives the same non-trivial linear relation over the principal ideal domain $\mathbb{Q}(\delta)[l^{\pm 1}]$ into which R embeds according to Lemma 4.1. Rescale the coefficients by a suitable power of $l-1$ to guarantee $\lambda_s \notin (l-1)\mathbb{Q}(\delta)[l^{\pm 1}]$ for some $s \in T$. Now $\mu(\lambda_s) \neq 0$ and $\pi(t) = \mu(\rho(t))$ for $t \in T$ (cf. Definitions 2.4), so $\sum_{t\in T} \mu(\lambda_t)\pi(t)$ is a non-trivial linear combination in $\mathbf{Br}(\mathrm{D}_n)$, that is equal to 0, contradicting the linear independence assumption on $\pi(T)$. $\square$

The following result will be applied with I the ideal of $\mathbf{Br}(\mathrm{D}_n)$ generated by the ideal Θ of $\mathbf{BrM}(\mathrm{D}_n)$ in the next section.

Proposition 4.3. *Let M be of type* ADE*. Suppose that I is an ideal of $\mathbf{Br}(M)$ such that $\mu^{-1}(I)$ is an ideal of $\mathbf{B}(M)$ that is free as an R-module with basis G consisting of monomials. Let T be a set of words in F_n whose image $\pi(T)$ under π is a basis of $\mathbf{Br}(M)/I$. If each word in F_n but outside $\pi^{-1}(I)$ can be reduced to an element of T, then $\rho(T) \cup G$ is a basis of $\mathbf{B}(M)$.*

Proof. We prove that $\rho(T)$ is a linear spanning set of $\mathbf{B}(M)/\mu^{-1}(I)$. If not, there is a word $\underline{a}$ in F_n such that $\rho(\underline{a})$ is not in the linear span of $\mu^{-1}(I)$ and $\rho(T)$. If $\pi(\underline{a}) \in I$, then $\rho(\underline{a}) \in \mu^{-1}(I)$, a contradiction. Hence $\underline{a} \notin \pi^{-1}(I)$, and, by assumption, $\underline{a} \rightsquigarrow \underline{b}$ for some $\underline{b} \in T$. Proposition 2.5(ii) implies that $\rho(\underline{a}) - \rho(\underline{b})$ is a linear combination of monomials in $\mathbf{B}(M)$ of height lower than $s = \mathrm{ht}(\underline{a})$. If $s = 0$, this means $\rho(\underline{a}) = \rho(\underline{b}) \in \rho(T)$. Otherwise $s > 0$ and we may assume, using induction on height, that monomials in $\mathbf{B}(M)$ outside $\mu^{-1}(I)$ of height lower than s are all in the linear span modulo $\mu^{-1}(I)$ of the elements in $\rho(T)$ of height lower than s. Then the right hand side in the expression of $\rho(\underline{a}) - \rho(\underline{b})$ as a linear combination of monomials of lower height is in the linear span of $\rho(T)$ and $\mu^{-1}(I)$. Consequently, $\rho(\underline{a})$ is in the same linear span, a contradiction. We have shown that $\mathbf{B}(M)/\mu^{-1}(I)$ is spanned by $\rho(T)$. The proposition now follows from Lemma 4.2 and the freeness assumptions on $\mu^{-1}(I)$. $\square$

5. Semisimplicity

In this section we prove Theorem 1.1. By Lemma 4.2, the rank of $\mathbf{B}(\mathrm{D}_n)$ is at least $\dim(\mathbf{Br}(\mathrm{D}_n))$, which by [3, Theorem 1.1] equals $(2^n+1)n!! - (2^{n-1}+1)n!$.

We will exhibit a set T of words in F_n, or rather $F_n/\leftrightsquigarrow$, with the property that its image under π is a basis of $\mathbf{Br}(\mathrm{D}_n)$ and such that any $\underline{a} \in F_n$ can be reduced to an element of T. By Proposition 2.5, this will suffice to give an upper bound for the rank of $\mathbf{B}(\mathrm{D}_n)$ and, by Lemma 4.2, $\rho(T)$ will be a basis. The set T will actually only be used for the quotient of the algebra by the ideal Θ' of $\mathbf{B}(\mathrm{D}_n)$ generated by e_1e_2. The image of Θ' under μ is denoted Θ''; it is the ideal of $\mathbf{Br}(\mathrm{D}_n)$ generated by e_1e_2 and so coincides with the ideal of $\mathbf{Br}(\mathrm{D}_n)$ generated by Θ of Lemma 2.8.

We first deal with Θ'. As shown in [4, Section 7.1], the elements g_1 and g_2 of $\mathbf{B}(\mathrm{D}_n)$ act the same by left and by right multiplication on all elements in Θ', and so Θ' is a homomorphic image of the ideal of $\mathbf{B}(\mathrm{A}_{n-1})$ generated by e_1. We claim that the ideal Θ'' of $\mathbf{Br}(\mathrm{D}_n)$ is free and has rank $n!! - n!$ over $\mathbb{Z}[\delta^{\pm1}]$. By [3, Lemma 3.3], cf. Theorem 3.2 and Lemma 2.8, Θ'' can be identified with the linear span of all elements of the form ue_Xzv^{-1} where $X \in \mathcal{A}$ is the maximal element of a W-orbit in $\mathcal{A}$ and contains the orthogonal mate of each root in X, $u, v^{-1} \in D_X$, and $z \in C_{WX}$. These elements are linearly independent in $\mathbf{Br}(\mathrm{D}_n)$ and are $n!! - n!$ in number, as is proved in [3, Corollary 5.5]. This establishes that Θ'' is as claimed.

According to Lemma 4.2, the rank of Θ' in $\mathbf{B}(\mathrm{D}_n)$ must be at least the rank of Θ''. Its rank cannot be more than the rank of the ideal in $\mathbf{B}(\mathrm{A}_{n-1})$ generated by e_1, which is $\dim(\mathbf{B}(\mathrm{A}_{n-1})) - |W(\mathrm{A}_{n-1})| = n!! - n!$ by [16]. From Lemma 4.2 it follows that Θ' is free of rank $n!! - n!$. Moreover, any set T_0 of words of minimal height in F_n corresponding to a basis of Θ'' consisting of monomials in $\Theta = \Theta'' \cap \mathbf{BrM}(\mathrm{D}_n)$ will work as an appropriate part of T corresponding to Θ', that is, each word in F_n representing an element of Θ can be reduced to an element of T_0, and the elements $\pi(t)$ for $t \in T_0$ are a basis of Θ''. In view of Proposition 4.3, with $I = 0$, $G = \emptyset$, $T = T_0$, and $M = \mathrm{A}_{n-1}$, the set $\rho(T_0)$ is a basis of Θ'.

For the remainder of the proof of the first statement of Theorem 1.1, let T_1 be the set of words in $F_n/\leftrightsquigarrow$ of the form $A_{B,n}zA^{\mathrm{op}}_{B',n}$ for $z \in U_{Y(t)}$ and B, $B' \in WY(t)$ for $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$ and similarly for $Y'(t)$ instead of $Y(t)$ in case $n = 2t$. By Corollary 3.18, each word in F_n outside $\pi^{-1}(\Theta)$ can be reduced to a word in T_1 and by [3, Proposition 4.9 and the proof of Theorem 1.1], $\pi(T_1)$ is a linearly independent spanning set of $\mathbf{Br}(\mathrm{D}_n)/\Theta''$. Now Proposition 4.3 applies with $I = \Theta''$, $G = \rho(T_0)$, $T = T_1$, and $M = \mathrm{D}_n$, so $\rho(T_1 \cup T_0)$ is a basis of $\mathbf{B}(\mathrm{D}_n)$.

To show that $\mathbf{B}(\mathrm{D}_n)$ tensored over $\mathbb{Q}(l,\delta)$ is semisimple we use the surjective ring homomorphism $\mu : \mathbf{B}(\mathrm{D}_n) \otimes_R \mathbb{Q}(\delta)[l^{\pm1}] \to \mathbf{Br}(\mathrm{D}_n)$ over $\mathbb{Q}(\delta)$; cf. Definitions 2.3. We know its image $\mathbf{Br}(\mathrm{D}_n)$ is semisimple by [3, Corollary 5.6] and so has no nilpotent left ideals. Suppose $\mathbf{B}(\mathrm{D}_n) \otimes_R \mathbb{Q}(\delta, l)$ has a nontrivial nilpotent ideal. Take a nonzero element of it expressed in the basis we have found. Multiply the element by a suitable polynomial in l so that all coefficients are in $\mathbb{Q}(\delta)[l^{\pm1}]$. As in the proof of Lemma 4.2, rescale the coefficients by a power of $l-1$ so that all coefficients remain in $\mathbb{Q}(\delta)[l^{\pm1}]$ but some coefficient λ_s lies outside $(l-1)\mathbb{Q}(\delta)[l^{\pm1}]$. The result is a nonzero nilpotent element in $\mathbf{B}(\mathrm{D}_n) \otimes \mathbb{Q}(\delta)[l^{\pm1}]$ with $\mu(\lambda_s) \neq 0$, so its image under π is a nonzero nilpotent element of $\mathbf{Br}(\mathrm{D}_n)$. Furthermore, any multiple is nilpotent both in $\mathbf{B}(\mathrm{D}_n) \otimes \mathbb{Q}(\delta, l)$ and in $\mathbf{Br}(\mathrm{D}_n)$ and so generates a nontrivial nilpotent ideal of $\mathbf{Br}(\mathrm{D}_n)$, a contradiction. This completes the proof of Theorem 1.1.

We will need the elements $A_{B,n}$ of F_n introduced in Notation 3.16. These elements were defined up to homogeneous equivalence. Since different elements from the homogeneous class of $A_{B,n}$ may give different elements in $\mathbf{B}(\mathrm{D}_n)$, see Remark 3.5, we need to select a particular element in each class.

Notation 5.1. For each $B \in WY(t)$, we take $A_{B,n}$ to be a specific word in F_n from its homogeneous equivalence class in F_n and write $b_{B,n} = \rho(a_{B,n})$ for its image in $\mathbf{B}(\mathrm{D}_n)$ under ρ.

Corollary 5.2. *For $n \geq 4$, there is a basis of $\mathbf{B}(\mathrm{D}_n)$ consisting of a basis of Θ' and elements of the form $b_{B,n}\rho(z)b_{B',n}$ for $(B, B', z) \in \bigcup_t Y(t) \times Y(t) \times U_{Y(t)} \cup Y'(n/2) \times Y'(n/2) \times U_{Y(n/2)}$.*

Remark 5.3. A consequence of Theorem 1.1 is that natural subalgebras generated by $\{g_i, e_i \mid i \in K\}$ for K a set of nodes of M have the usual desired subalgebra structure, that is, are naturally isomorphic to the BMW algebra whose type is the restriction of M to K. In particular, the subalgebra generated by $\{g_i, e_i \mid 2 \leq i \leq n\}$ is the full $\mathbf{B}(\mathrm{A}_{n-1})$ rather than a proper homomorphic image. The same applies to the algebra generated by all g_i, e_i for $i \leq n-1$ which is $\mathbf{B}(\mathrm{D}_{n-1})$ and not a proper image.

Remark 5.4. The results imply that the generalized Temperley–Lieb algebra of type D_n, cf. [9, 12, 14], embeds in $\mathbf{Br}(\mathrm{D}_n)$. The elements e_i either in $\mathbf{B}(\mathrm{D}_n)$ or in $\mathbf{Br}(\mathrm{D}_n)$ commute for $i \not\sim j$ by (HCee). For $i \sim j$, we have $e_ie_je_i = e_i$ by (HNeee). Also, $e_i^2 = \delta e_i$ by (HSee). The free algebra on $e_1, \ldots, e_n$ with this presentation over $\mathbb{Z}[\delta^{\pm 1}]$ is called the (generalized) Temperley–Lieb algebra of type D_n; we will denote it by $\mathbf{TL}(\mathrm{D}_n)$. The subalgebra generated by $e_1, \ldots, e_n$ in $\mathbf{Br}(\mathrm{D}_n)$ is a homomorphic image of $\mathbf{TL}(\mathrm{D}_n)$; the subalgebra of $\mathbf{B}(\mathrm{D}_n)$ generated by these elements is a homomorphic image of $\mathbf{TL}(\mathrm{D}_n) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} R$. The words in F_n corresponding to generators for these subalgebras consist solely of the symbols $e_1, \ldots, e_n, \delta$ and so are of height 0.

In [14, Theorem 4.2 and Lemma 6.5], a description of a generating set for the Temperley–Lieb algebra is given in terms of diagrams. (In [6] diagrams such as these will be introduced for the full algebra $\mathbf{Br}(\mathrm{D}_n)$.) The diagrams are in bijective correspondence with monomials of $\mathbf{TL}(\mathrm{D}_n)$ in the e_i. There is an ideal in this description that is linearly spanned by diagrams of type 1 in the terminology [14]. It has rank $\frac{1}{n+1}\binom{2n}{n} - 1$, the n-th Catalan number minus 1. Each monomial in this ideal can be written as a multiple of the diagram for e_1e_2. There is one more set of diagrams, called of type 2 in [14]. The number of these is $\frac{1}{2}\binom{2n}{n}$; see [14, Lemma 6.5]. Their linear span is not a subalgebra, but it is a complement in $\mathbf{TL}(\mathrm{D}_n)$ of the ideal spanned by the diagrams of type 1. So the rank of $\mathbf{TL}(\mathrm{D}_n)$ is equal to $\frac{n+3}{2n+2}\binom{2n}{n} - 1$.

Now consider the images of the Temperley–Lieb monomials in $\mathbf{Br}(\mathrm{D}_n)$. Those of type 1 are in the ideal Θ'', which is isomorphic to the ideal in $\mathbf{B}(\mathrm{A}_{n-1})$ generated by e_1 (cf. the description of Θ'' in the proof above and Remark 5.3). The subalgebra of $\mathbf{B}(\mathrm{A}_{n-1})$ generated by the monomials of height 0 is isomorphic to the Temperley–Lieb algebra, $\mathbf{TL}(\mathrm{A}_{n-1})$, and its intersection with the ideal generated by e_1 is free of rank $\dim(\mathbf{TL}(\mathrm{A}_{n-1})) - \dim(\mathbf{TL}(\mathrm{A}_{n-1})/(e_1)) = \frac{1}{n+1}\binom{2n}{n} - 1$. These monomials have already been accounted for in Θ''.

What remains of the image in $\mathbf{Br}(\mathrm{D}_n)/\Theta''$ of the Temperley–Lieb algebra $\mathbf{TL}(\mathrm{D}_n)$ is the span of the monomials coming from diagrams of type 2. For these, we

use Corollary 3.18, which states that the words representing a given monomial of $\mathbf{Br}(\mathrm{D}_n)$ not in Θ'' can all be reduced to a unique one. But for reducing words of height 0, only relations (HCee), (HNeee), and (HSee) can be used, so if two words of height 0 giving the same pair reduce to the same element, they will represent the same element in $\mathbf{TL}(\mathrm{D}_n)$. This means that their number equals the number of monomials in $\mathbf{TL}(\mathrm{D}_n)$ outside the ideal generated by e_1e_2, viz. $\frac{1}{2}\binom{2n}{n}$. Therefore, the homomorphic image of $\mathbf{TL}(\mathrm{D}_n)$ in $\mathbf{Br}(\mathrm{D}_n)$ has the rank of $\mathbf{TL}(\mathrm{D}_n)$ and so is isomorphic to it. By Proposition 2.5(ii), the same holds for $\mathbf{B}(\mathrm{D}_n)$ when tensored with R.

Remark 5.5. By use of μ and the Tits Deformation Theorem, see [2, IV.2, exercice 26] or [18, Lemma 85], it can be shown that the irreducible degrees associated to $\mathbf{B}(\mathrm{D}_n)$ are the same as for $\mathbf{Br}(\mathrm{D}_n)$. This can also be shown by use of Theorem 3.17 for representations with Θ' in the kernel as in [3] and for the others from the connection of Θ' to $\mathbf{B}(\mathrm{A}_{n-1})$ as in the proof of Theorem 1.1.

6. Cellularity

Let S be a commutative algebra over R. In this section we prove Theorem 1.2, which states that $\mathbf{B}(\mathrm{D}_n)\otimes_R S$ is cellular in the sense of Graham–Lehrer [13, Definition 1.1] provided S contains an inverse to 2. Recall from [13] that an associative algebra A over a commutative ring S is cellular if there is a quadruple $(\Lambda, T, C, *)$ satisfying the following three conditions.

(C1) Λ is a finite partially ordered set. Associated to each $\lambda \in \Lambda$, there is a finite set $T(\lambda)$. Also, C is an injective map

$$\coprod_{\lambda\in\Lambda} T(\lambda)\times T(\lambda) \to A$$

whose image is an S-basis of A.

(C2) The map $* : A \to A$ is an S-linear anti-involution such that $C(x,y)^* = C(y,x)$ whenever $x, y \in T(\lambda)$ for some $\lambda \in \Lambda$.

(C3) If $\lambda \in \Lambda$ and $x, y \in T(\lambda)$, then, for any element $a \in A$,

$$aC(x,y) \equiv \sum_{u\in T(\lambda)} r_a(u,x)C(u,y) \mod A_{<\lambda},$$

where $r_a(u,x) \in S$ is independent of y and where $A_{<\lambda}$ is the S-submodule of A spanned by $\{C(x',y') \mid x', y' \in T(\mu) \text{ for } \mu < \lambda\}$.

Such a quadruple $(\Lambda, T, C, *)$ is called a *cell datum* for A.
Now let S be a commutative algebra over R with $2^{-1} \in S$. We introduce a quadruple $(\Lambda, T, C, *)$ and prove that it is a cell datum for $A = \mathbf{B}(\mathrm{D}_n)\otimes_R S$. The map $*$ on A will be the opposition map $\cdot^{\mathrm{op}}$ of Notation 3.1. Before describing the other three components, we will relate the subalgebras of A generated by monomials corresponding to the elements of $U_{Y(t)}$, defined in Proposition 3.9(iii), to Hecke algebras. For this purpose we need a version of Proposition 3.9 that applies to A rather than $\mathbf{BrM}(\mathrm{D}_n)$. This requires a version of Lemma 3.7 for $\mathbf{B}(\mathrm{D}_n)$ rather than $F_n/\leftrightsquigarrow$.

Lemma 6.1. *For $n \geq 3$, the monomials $\hat{z}_n^* = \rho(\underline{z}_n^*)$ have height 1 and satisfy the following equations, where $g_n^* = \rho(r_n^*)$.*

(i) $g_n^* e_n = \hat{z}_n^* = e_n g_n^*$.

(ii) $\hat{z}_n^* = e_{n,3}g_2e_1e_{3,n}$.
(iii) For $n \geq 4$ and $i \in \{1,\dots,n-2\}$, $e_i\hat{z}_n^* = \hat{z}_i^*e_n = e_n\hat{z}_i^*$ and $g_i\hat{z}_n^* = \hat{z}_n^*g_i$.
(iv) $\hat{z}_n^*e_{n-2} = e_n\hat{z}_{n-2}^*$ and $e_n\hat{z}_n^* = \hat{z}_n^*e_n = \delta\hat{z}_n^*$.
(v) $(\hat{z}_n^*)^2 \in \delta e_n - m\delta\hat{z}_n^* + \Theta'$.

Proof. By definition, there is only one factor g_i in $\hat{z}_n^*$ and so its height is at most 1. A look at $\mu(\hat{z}_n^*) = \pi(\underline{z}_n^*)$ in $\mathbf{BrM}(\mathrm{D}_n)$ shows that the height must be equal to 1.
(i). The proof is similar to the one of Lemma 3.7(i); note that the relations (RNrre) for $\mathbf{B}(\mathrm{D}_n)$ are also binomial.
(ii). Again, the only relation used in the proof of Lemma 3.7(ii) is (HTeere), which is also binomial for $\mathbf{B}(\mathrm{D}_n)$.
(iii). Let $i \in \{2,\dots,n-2\}$. The relation $e_i\hat{z}_n^* = e_n\hat{z}_i^*$ can be derived from the definition of $e_{k,n}$, and the binomial relations (HCee) and (HNeee), as in the proof of Lemma 3.7.
The proof of $g_i\hat{z}_n^* = \hat{z}_n^*g_i$ is a bit more involved. By (HCer), (HNree), and (HNeer),

$$\begin{aligned} g_i\hat{z}_n^* &= e_{n,i+2}g_ie_{i+1,2}g_1e_{3,n} = e_{n,i+2}g_i^2g_{i+1}e_{i,2}g_1e_{3,n} \\ &= e_{n,i+2}g_{i+1}e_{i,2}g_1e_{3,n} - me_{n,i+2}g_ig_{i+1}e_{i,2}g_1e_{3,n} \\ &\qquad\qquad +ml^{-1}e_{n,i+2}e_ig_{i+1}e_{i,2}g_1e_{3,n} \\ &= e_{n,i+2}e_{i+1}g_{i+2}^{-1}e_{i,2}g_1e_{3,n} - me_{n,2}g_1e_{3,n} + me_{n,i+2}e_{i,2}g_1e_{3,n} \\ &= e_{n,i+1}g_{i+2}^{-1}e_{i,2}g_1e_{3,n} - m\hat{z}_n^* + me_{i,2}g_1e_{n,i+2}e_{3,n} \\ &= e_{n,2}g_{i+2}^{-1}g_1e_{3,n} - m\hat{z}_n^* + me_{i,2}g_1e_{n,i}e_n \\ &= e_{n,2}g_{i+2}^{-1}g_1e_{3,n} - m\hat{z}_n^* + m\hat{z}_i^*e_n. \end{aligned}$$

Since each of the three summands is invariant under opposition, (observe that $(\hat{z}_n^*)^{\mathrm{op}} = \hat{z}_n^*$ follows from (i)), so is $g_i\hat{z}_n^*$. This shows $g_i\hat{z}_n^* = (g_i\hat{z}_n^*)^{\mathrm{op}} = (\hat{z}_n^*)^{\mathrm{op}}g_i = \hat{z}_n^*g_i$.
The case $i=1$ is notationally different but can be done the same way as $i=2$.
(iv). By (iii) with $i = n-2$ we have $e_{n-2}\hat{z}_n^* = \hat{z}_{n-2}^*e_n = e_n\hat{z}_{n-2}^*$. Taking images under $\cdot^{\mathrm{op}}$ and using opposition invariance of $\hat{z}_n^*$, we find $\hat{z}_n^*e_{n-2} = (e_{n-2}\hat{z}_n^*)^{\mathrm{op}} = (\hat{z}_{n-2}^*e_n)^{\mathrm{op}} = e_n\hat{z}_{n-2}^*$, as required for the first equation. The second chain of equations is a direct consequence of (RSee).
(v). For $n \geq 3$, by (HSee), (HCer), (HNeee), and (RSrr),

$$\begin{aligned} (\hat{z}_n^*)^2 &= e_{n,3}g_1e_2e_{3,n}e_{n,3}g_1e_2e_{3,n} = e_{n,3}g_1e_2e_3g_1e_2e_{3,n}\delta \\ &= \delta e_{n,3}g_1e_2e_3e_2g_1e_{3,n} = \delta e_{n,3}g_1e_2g_1e_{3,n} = \delta e_{n,3}g_1^2e_2e_{3,n} \\ &= \delta e_{n,3}(1 - mg_1 + ml^{-1}e_1)e_{2,n} \\ &= \delta e_{n,3}e_{2,n} - \delta me_{n,3}g_1e_{2,n} + \delta ml^{-1}e_{n,3}e_1e_2e_{3,n} \\ &\in \delta e_n - \delta m\hat{z}_n^* + \Theta'. \end{aligned}$$

The cases $n=1$ and $n=2$ are easily proved separately. □

Definition 6.2. For $t \in \{0,\dots,\lfloor n/2\rfloor\}$, we write J_{t+1} to denote the ideal of A generated by Θ', $e_{Y(t')}$ for all $t' > t$, and $e_{Y'(n/2)}$ if n is even and $n > 2t$. So J_1 is the ideal generated by all e_i.

Recall the words $\underline{s}_i$ $(0 \leq i \leq n-2t)$ from (2). We now state the $\mathbf{B}(\mathrm{D}_n)$-variant of Proposition 3.9.

Proposition 6.3. *Let $t \in \{0,\ldots,\lfloor n/2\rfloor\}$. The monomials $\hat{s}_i = \rho(\underline{s}_i)$, for i a node of $C_{Y(t)}$, satisfy the following relations.*

(i) The element $\rho(1_t)$ acts as an identity element on the $\hat{s}_i$, that is, $\rho(1_t)\hat{s}_i = \hat{s}_i$ and $\hat{s}_i\rho(1_t) = \hat{s}_i$, while $\rho(1_t)^2 = \rho(1_t)$. Moreover, the $\hat{s}_i$ satisfy the braid relations (HCrr) and (HNrrr) of Table 1 with g_i replaced by $\hat{s}_i$ and 1 by $\rho(1_t)$.

(ii) Each monomial $\hat{s}_i$ satisfies the quadratic Hecke algebra relation modulo the ideal J_{t+1}, that is, $\hat{s}_i^2 + m\hat{s}_i - \rho(1_t) \in J_{t+1}$.

Proof. (i). The relations involving $\rho(1_t)$ are easily derived from Lemma 6.1. Note the resemblance with the proof of Proposition 3.9.

Use of (RSrr), (HCrr), and (HNrrr) gives the relations not involving $\hat{s}_0$. It remains to verify the commuting of $\hat{s}_0$ with $\hat{s}_i$ for $i \in \{1,\ldots,n-2t\}$. By Lemma 6.1(iii) $g_i\hat{z}_n^* = \hat{z}_n^* g_i$. This gives $\hat{s}_0\hat{s}_i = z_n^*\delta^{-1}1_t g_i 1_t = z_n^* g_i \delta^{-1}1_t = g_i 1_t z_n^*\delta^{-1}1_t = \hat{s}_i\hat{s}_0$.

(ii). For $i \in \{1,\ldots,n-2t\}$, we have $\hat{s}_i^2 = g_i 1_t g_i 1_t = g_i^2 1_t = (1 - mg_i + ml^{-1}e_i)1_t = 1_t - m\hat{s}_i + ml^{-1}e_{Y(t)\cup\{i\}}\delta^{-1-t}$. Here $e_{Y(t)\cup\{i\}}$ is in J_{t+1}, so $\hat{s}_i^2 + m\hat{s}_i - 1_t \in J_{t+1}$. As, for $\hat{s}_0^2$, by Lemma 6.1(iv), $\hat{z}_n^* e_{Y(t)} = e_{Y(t)}\hat{z}_{n-2t+2}^*$ and $e_{Y(t)}\hat{z}_n^* = \hat{z}_{n-2t+2}^* e_{Y(t)}$, so, by Lemma 6.1(iii),(iv),(v), as $\Theta' \subseteq J_{t+1}$,

$$\begin{aligned}
\hat{s}_0^2 &= \hat{z}_n^* e_{Y(t)}\hat{z}_n^* e_{Y(t)}\delta^{-2t-2} = \hat{z}_n^* e_{Y(t)} e_{Y(t)}\hat{z}_n^* e_{Y(t)}\delta^{-3t-2}\\
&= e_{Y(t)}(\hat{z}_{n-2t+2}^*)^2 e_{Y(t)}^2\delta^{-3t-2}\\
&\in e_{Y(t)}(e_{n-2t+2} - m\hat{z}_{n-2t+2}^*)e_{Y(t)}\delta^{-2t-1} + J_{t+1}\\
&= e_{Y(t)}e_{n-2t+2}e_{Y(t)}\delta^{-2t-1} - me_{Y(t)}\hat{z}_{n-2t+2}^* e_{Y(t)}\delta^{-2t-1} + J_{t+1}\\
&= \rho(1_t) - m\hat{z}_n^* e_{Y(t)}e_{Y(t)}\delta^{-2t-1} + J_{t+1}\\
&= \rho(1_t) - m\hat{z}_n^* e_{Y(t)}\delta^{-t-1} + J_{t+1}\\
&= \rho(1_t) - m\hat{s}_0 + J_{t+1}.
\end{aligned}$$

□

We will next exploit the elements $b_{B,n}$ of Notation 5.1. Recall from Proposition 3.9(iii) the definition of $U_{Y(t)}$.

Corollary 6.4. *For $t \in \{0,\ldots,\lfloor n/2\rfloor\}$, the linear span $H_{Y(t)}$ of $\rho(U_{Y(t)})$ in A satisfies the following properties.*

(i) The linear subspace $H_{Y(t)} + J_{t+1}$ is a subalgebra of A whose quotient algebra mod J_{t+1} is isomorphic to the Hecke algebra of type $C_{Y(t)}$. Moreover, $\rho(U_{Y(t)})$ is a basis of $H_{Y(t)}$.

(ii) For each $i \in \{1,\ldots,n\}$ and $B \in WY(t)$, we have $g_i b_{B,n} e_{Y(t)} \in b_{r_iB,n}H_{Y(t)} + J_{t+1}$ and $e_i b_{B,n} e_{Y(t)} \in b_{B'',n}H_{Y(t)} + J_{t+1}$ for some $B'' \in WY(t)$.

(iii) $H_{Y(t)}$ is invariant under opposition.

If n is even, the similarly defined linear span $H_{Y'(n/2)}$ equals $S1'_{n/2}$ and satisfies the same properties.

Proof. If $\underline{a} \in U_{Y(t)}$ is a minimal expression in the $\underline{s}_i$ of the element $\pi(\underline{a}) \in W(C_{Y(t)})$, then, as a consequence of Lemma 3.6 and the relations established in Proposition 6.3(i), $\rho(\underline{a})$ depends only on $\pi(\underline{a})$ and not on the choice of the minimal expression.

(i). By the above and Proposition 6.3(ii), the spanning set $\rho(U_{Y(t)})$ of $H_{Y(t)}$ has size at most $|W(C_{Y(t)})|$. Due to Corollary 5.2 there is no collapse, so the spanning

set has size equal to $|W(C_{Y(t)})|$ and is a basis of $H_{Y(t)}$. By Proposition 6.3, the linear subspace $H_{Y(t)} + J_{t+1}$ is closed under multiplication and satisfies the Hecke algebra defining relations mod J_{t+1} on the generators $\hat{s}_i$ $(0 \le i \le n-2t)$. In particular, $(H_{Y(t)} + J_{t+1})/J_{t+1}$ is a quotient of the Hecke algebra of type $C_{Y(t)}$. But, its rank is equal to $|W(C_{Y(t)})|$, which is the Hecke algebra dimension, and so $(H_{Y(t)} + J_{t+1})/J_{t+1}$ is isomorphic to the Hecke algebra of type $C_{Y(t)}$.

(ii). In view of Corollary 5.2 and Proposition 2.7, $b_{B,n}H_{Y,t}b_{B',n} + J_{t+1}$ is the linear span of J_{t+1} and all monomials x in $\mathbf{B}(\mathrm{D}_n)$ such that $\mu(x)(\emptyset) = B$ and $\mu(x)^{\mathrm{op}}(\emptyset) = B'$. But $x = g_i b_{B,n} e_{Y,t}$ satisfies $\mu(x)(\emptyset) = r_i B$ and $\mu(x)^{\mathrm{op}}(\emptyset) = Y(t)$, so $g_i b_{B,n} e_{Y,t} \in b_{B,n} H_{Y,t} + J_{t+1}$.
Similarly, $\mu(e_i b_{B,n} e_{Y(t)})(\emptyset) = \mu(e_i b_{B,n})Y(t) = e_i B$ always contains a member B'', say, of WB, and $\mu(e_i b_{B,n} e_{Y(t)})^{\mathrm{op}}(\emptyset) = e_{Y(t)}(\pi(A_{B,n}))^{\mathrm{op}}\{\alpha_i\}$ contains $Y(t)$, so $e_i b_{B,n} e_{Y(t)} \in b_{B'',n} H_{Y(t)} + J_{t+1}$.

(iii). It is readily verified that each $\hat{s}_i$ is fixed under opposition. As the opposite of a minimal expression in the $\hat{s}_i$ is again a minimal expression, $\rho(U_{Y(t)})$ is invariant under opposition. Hence, so is $H_{Y(t)}$. □

We now give the cell datum for $A = \mathbf{B}(\mathrm{D}_n)_R \otimes S$. View A_{n-1} as the subdiagram of D_n on the nodes $2, \ldots, n$. As an algebra over S, the ideal Θ' of A generated by e_1e_2 is isomorphic to the ideal of $\mathbf{B}(\mathrm{A}_{n-1}) \otimes_R S$ generated by e_2. This algebra itself is cellular as $\mathbf{B}(\mathrm{A}_{n-1})$ is cellular by [21, Theorem 3.11] and it is one of the ideals given by the cellular structure. In fact, it is the one indexed by all partitions of $n-2t$ for $1 \le t \le \lfloor n/2 \rfloor$. Let $(\Lambda_\theta, T_\theta, C_\theta, *_\theta)$ be the cell datum for Θ'. It is clear from [21, Theorem 3.11] that $*_\theta$ coincides with the restriction to Θ' of the map $\cdot^{\mathrm{op}}$. Moreover, the elements $g_1 - g_2$ and $e_1 - e_2$ are in the kernel of the action of A on Θ' by left multiplication, as well as by right multiplication.
For $0 \le t \le \lfloor n/2 \rfloor$ we let $(\Lambda_t, T_t, C_t, *_t)$ be the cell datum for the Hecke algebra $H_{Y(t)}$ mod J_{t+1} of type $C_{Y(t)}$ (see Corollary 6.4(i)) with $*_t$ the restriction to $H_{Y(t)}$ of $\cdot^{\mathrm{op}}$. If $n = 2t$, there is another copy needed which we denote $(\Lambda'_{n/2}, T'_{n/2}, C'_{n/2}, *'_{n/2})$; it corresponds to the admissible set $Y'(n/2)$. By [11], these cell data are known to exist provided $\frac{1}{2} \in S$. We take the values of C_t to be in $H_{Y(t)}$.
The poset Λ is the disjoint union of the posets Λ_t of the cell data for the various Hecke algebras $H_{Y(t)}$ mod J_{t+1}, as well as $\Lambda'_{n/2}$ if n is even. We make Λ into a poset as follows. For a fixed t or θ it is already a poset, and we keep the same partial order. Furthermore, any element of Λ_t is greater than any element of Λ_s if $t < s$. In particular the elements of Λ_0 are greater than the elements of Λ_t for any $t \ge 1$. Moreover, if n is even, any element of $\Lambda'_{n/2}$ is smaller than any element of Λ_t for $t < n/2$. Finally, we decree that any element of Λ_θ is smaller than any element of Λ_t $(0 \le t \le n/2)$ or $\Lambda'_{n/2}$.

Let $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$. For $\lambda \in \Lambda_t$, we set $T(\lambda) = WY(t) \times T_t(\lambda)$ and, if n is even, for $\lambda \in \Lambda'_{n/2}$, we set $T(\lambda) = WY'(n/2) \times T'_{n/2}(\lambda)$. For $\lambda \in \Lambda_\theta$, we set $T(\lambda) = T_\theta(\lambda)$. This determines T.

We define C as follows. For $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$, $\lambda \in \Lambda_t$, and $(B, x), (B', y) \in T(\lambda)$, we have
$$C\big((B,x),(B',y)\big) = b_{B,n} C_t(x,y) b^{\mathrm{op}}_{B',n}.$$
Similarly on $\Lambda'_{n/2} \times \Lambda'_{n/2}$. For $\lambda \in \Lambda_\theta$, the map C on $T(\lambda) \times T(\lambda)$ is just C_θ.

Since we have already defined $*$ by the opposition map, this concludes the definition of $(\Lambda, T, C, *)$. We next verify the conditions (C1), (C2), (C3).

(C1). The map C has been chosen so that its image is an S-basis of Θ' (the image of C_θ), joint with the set of elements $b_{B,n}C_t(x,y)b_{B',n}^{\mathrm{op}}$ for $B, B' \in WY(t)$ and $C_t(x,y)$ running through a basis of $H_{Y(t)}$, and $b_{B,n}C'_{n/2}(x,y)b_{B',n}^{\mathrm{op}}$ for $B, B' \in WY'(n/2)$ and $C'_{n/2}(x,y)$ running through a basis of $H_{Y'(n/2)}$. By Corollary 5.2, this implies that the image of C is a basis of A. Injectivity of C follows from injectivity of C_θ, C_t $(0 \le t \le n/2)$, $C'_{n/2}$ if n is even, and Theorem 1.1, which guarantees that no collapses of dimensions of the individual parts occur.

(C2). Clearly, $*$ is an S-linear anti-involution. Let $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$, $\lambda \in \Lambda_t$, and $(B,x),(B',y) \in T(\lambda)$. Then $(b_{B,n}C_t(x,y)b_{B',n}^{\mathrm{op}})^{\mathrm{op}} = b_{B',n}C_t(x,y)^{\mathrm{op}}b_{B,n}^{\mathrm{op}}$, so, in order to establish $\big(C((B,x),(B',y))\big)^* = C((B',y),(B,x))$, it suffices to verify that $C_t(x,y)^{\mathrm{op}}$ coincides with $C_t(y,x)$. Now $*_t$ on $H_Y(t)$ mod J_{t+1} coincides with opposition, so modulo J_{t+1} we have $C_t(x,y)^{\mathrm{op}} = C_t(x,y)^{*_t} = C_t(y,x)$ by the cellularity of $(\Lambda_t, T_t, C_t, *_t)$. On the other hand, as $H_{Y(t)}$ is invariant under opposition, see Corollary 6.4(iii), and contains the values of C_t, it contains $C_t(x,y)^{\mathrm{op}} - C_t(y,x)$, so $C_t(x,y)^{\mathrm{op}} - C_t(y,x) \in H_{Y(t)} \cap J_{t+1} = \{0\}$, whence $C_t(x,y)^{\mathrm{op}} = C_t(y,x)$, as required.
The case of $\lambda \in \Lambda''_{n/2}$ for n even is similar. If $\lambda \in \Lambda_\theta$ and $x, y \in T(\lambda)$, then $C(x,y)^* = C(y,x)$ is immediate from the cellularity of $(\Lambda_\theta, T_\theta, C_\theta, *_\theta)$.

(C3). Let $\lambda \in \Lambda_t$ and $(B,x),(B',y) \in T(\lambda)$. Fix $i \in \{1, \ldots, n\}$. It clearly suffices to prove the formulas for a running over the generators g_i and e_i of $\mathbf{B}(\mathrm{D}_n)$.
By choice of C_t, we have $C_t(x,y) \in H_{Y(t)}$, and, see Proposition 6.3(i), $C_t(x,y) = \rho(1_t)C_t(x,y)$. According to Corollary 6.4(iii), there is $z_{B,i} \in H_{Y(t)}$, depending only on B and i, such that $g_i b_{B,n}\rho(1_t) \in b_{g_iB,n}z_{B,i} + J_{t+1}$. As $(\Lambda_t, T_t, C_t, *_t)$ is a cell datum for $H_{Y(t)}$ mod J_{t+1}, there are $\nu_i(u,B,x) \in S$, independent of B' and y, for each $u \in T_t(\lambda)$ such that

$$z_{B,i}C_t(x,y) \quad \in \quad \sum_{u \in T_t(\lambda)} \nu_i(u,B,x)C_t(u,y) + (H_{Y(t)})_{<\lambda} + J_{t+1}.$$

Since both $(H_{Y(t)})_{<\lambda}$ and J_{t+1} are contained in $A_{<\lambda}$, we find

$$\begin{aligned} g_iC((B,x),(B',y)) &= g_ib_{B,n}\rho(1_t)C_t(x,y)b_{B',n}^{\mathrm{op}} \\ &\in b_{g_iB,n}z_{B,i}C_t(x,y)b_{B',n}^{\mathrm{op}} + A_{<\lambda} \\ &= \sum_{u \in T_t(\lambda)} \nu_i(u,B,x)b_{g_iB,n}C_t(u,y)b_{B',n}^{\mathrm{op}} + A_{<\lambda} \\ &= \sum_{u \in T_t(\lambda)} \nu_i(u,B,x)C((g_iB,u),(B',y)) + A_{<\lambda} \end{aligned}$$

as required.
Rewriting (RSrr) to $e_i = lm^{-1}(g_i^2 + mg_i - 1)$, we see that, if $m^{-1} \in S$, the proper behavior of the cell data under left multiplication by e_i is taken care of by the above formulae for g_i. A proof in full generality can be given that is similar to the above proof for g_i.
For $\lambda \in \Lambda_\theta$, the formulas are straight from those for Θ' as $g_1a = g_2a$ and $e_1a = e_2a$ for each $a \in \Theta'$.

This establishes that $(\Lambda, T, C, *)$ is a cell datum for A and so completes the proof of Theorem 1.2.

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